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Gluon polarization tensor in color magnetic background

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Abstract. In SU(2) gluodynamics we calculate the gluon polarization tensor in an Abelian homogeneous magnetic field in one-loop order in the Lorentz background field gauge. It turned out to be non-transversal and consisting of ten tensor structures and corresponding form factors – four in color neutral and six in color charged sector. Seven tensor structures are transversal, three are not. The non-transversal parts are obtained by explicit calculation. We represent the form factors in terms of double parametric integrals which can be computed numerically. Some examples are provided and possible applications are discussed.

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1 Introduction

The investigation of the deconfinement phase in QCD has attracted considerable interest in the past years. The current understanding is that this phase is not simply a non-interacting gas of quarks and gluons as assumed earlier but a non-trivial state of these fields. In particular, from lattice calculations [1] and from perturbative daisy resummations [2,3] it was found that a chromomagnetic field of order $gB \sim g^4T^2$, where g is the gauge coupling, B is the magnetic field strength and T is the temperature, is likely to appear spontaneously. These results are of importance not only for the QCD but also for the possibility to explain the origin of the cosmological magnetic field in the early universe [4].

In the perturbative approach because of the known infrared problems it is necessary to make resummations, on the daisy level, on the super-daisy level and, if possible, beyond. In any case the knowledge of the gluon polarization tensor in a magnetic field, including also non-zero temperature and an A_0-condensate, is necessary.

The calculation of the polarization tensor in an external magnetic field has a long history, especially in QED. However in QCD much less is known. This is perhaps because in QCD it does not have a direct physical significance (except for the case of finite temperature for which it had been investigated in detail for example in [5]). Particular results had been obtained for example in [2,3,6–8]. Among them the most important was the observation that in the background of a magnetic field the "charged" gluons acquire a magnetic mass proportional to $g^2(gB)^{1/2}T$ [9], whereas the "neutral" ones corresponding to the "Abelian projection" remain massless [3]. In general, the question about the magnetic mass and whether it can cure the tachyonic instability is discussed.

However it is a common feature of all these calculations that the polarization tensor in a magnetic field had never been calculated in detail but only in some approximations and projections on physical states, i.e., on-shell. Nevertheless its structure off-shell is an indispensable prerequisite for higher order resummations like applying the second Legendre transform. Also it is of interest for the W-bosons in the electroweak theory and for the question how to define the gluon magnetic mass.

So in the present paper we investigate the gluon polarization tensor in SU(2) gluodynamics in detail off-shell in a homogeneous color magnetic background at zero temperature. We use the standard Lorentz background field gauge. Our first finding is that the tensor is not transversal, $p_\mu \Pi_{\mu\nu} \neq 0$. This fact is insofar remarkable as it is not in line with the common expectations. In fact its transversality in the presence of a magnetic field had never been shown. We find the non-transversal contributions from direct calculation. The structure of the polarization tensor is determined by the weaker condition $p_\mu \Pi_{\mu\nu} p_\nu = 0$. There are four independent tensor structures $T^{(i)}_{\mu\nu}$ for the color neutral polarization tensor and six for the charged one. Corrsponding to each tensor structure there is a corresponding form factor.

It should be mentioned that the color neutral component of the polarization tensor had been calculated repeatedly in Fujikawa gauge [2,3,10,11]. In that case the structure is much simpler, especially because the tensor is transversal. In addition it has only three tensor structures and accordingly only three form factors instead of four in background gauge (see below). But in this gauge the color charged components are much more complicated.

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In the next section we introduce the necessary notation and review the basic formulas. In Sects. 3 and 4 we derive the general structure and explicit formulas in the representation of parameter integrals for the neutral and for the charged polarization tensor. In Sect. 5 we discuss the renormalization and perform as an example some numerical calculations.

Throughout the paper we use Latin letters $a, b, \ldots = 1, 2, 3$ for the color indices and Greek letters $\lambda, \mu, \ldots = 1, \ldots, 4$ for the Lorentz indices. Summation over doubly appearing indices is assumed. All formulas are in the Euclidean formulation unless otherwise indicated. We put all constants including the coupling equal to unity.

2 Basic notation

In this section we collect the well known basic formulas for SU(2) gluodynamics to set up the notation which we will use. We work in the Euclidean version basically to avoid unnecessary signs and factors i. Dropping arguments and indices the Lagrangean is simply

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + \overline{\eta}\partial_\mu D_\mu \eta, \tag{1}$$

where ξ is the gauge parameter and η is the ghost field. The action is $S = \int \mathrm{d}x\ L$ and the generating functional of the Green functions is $Z = \int DA \exp(S)$. In the following we divide the gauge field $A_\mu^a(x)$ into background field $B_\mu^a(x)$ and quantum fluctuations $Q_\mu^a(x)$,

$$A_\mu^a(x) = B_\mu^a(x) + Q_\mu^a(x). \tag{2}$$

The covariant derivative depending on a field A is

$$D_\mu^{ab}[A] = \frac{\partial}{\partial x^\mu}\delta^{ab} + \epsilon^{acb}A_\mu^c(x) \tag{3}$$

and the field strength is

$$F_{\mu\nu}^a[A] = \frac{\partial}{\partial x^\mu}A_\nu^a(x) - \frac{\partial}{\partial x^\nu}A_\mu^a(x) + \epsilon^{abc}A_\mu^b(x)A_\nu^c(x) \tag{4}$$

and

$$[D_\mu[A], D_\nu[A]]^{ab} = \epsilon^{acb}F_{\mu\nu}^c[A] \tag{5}$$

holds. For the field splitted into background and quantum we note

$$\begin{aligned} &F_{\mu\nu}^a[B+Q] \\ &= F_{\mu\nu}^a[B] + D_\mu^{ab}[B]Q_\nu^b(x) - D_\nu^{ab}[B]Q_\mu^b(x) \\ &\quad + \epsilon^{abc}Q_\mu^b(x)Q_\nu^c(x). \end{aligned} \tag{6}$$

The square of it is

$$\begin{aligned} &-\frac{1}{4}\left(F_{\mu\nu}^a[B+Q]\right)^2 \\ &= -\frac{1}{4}\left(F_{\mu\nu}^a[B]\right)^2 + Q_\nu^a D_\mu^{ab}[B]F_{\mu\nu}^b[B] \\ &\quad -\frac{1}{2}Q_\mu^a K_{\mu\nu}^{ab}Q_\nu^b + \mathcal{M}_3 + \mathcal{M}_4. \end{aligned} \tag{7}$$

The second term in the RHS is linear in the quantum field and disappears if the background fulfills its equation of motion which will hold in our case of a constant background. The third term is quadratic in Q_μ^a and it defines the "free part" with the kernel

$$\begin{aligned} K_{\mu\nu}^{ab} = &-\delta_{\mu\nu}D_\rho^{ac}[B]D_\rho^{cb}[B] + D_\mu^{ac}[B]D_\nu^{cb}[B] \\ &-2\epsilon^{acb}F_{\mu\nu}^c[B]. \end{aligned} \tag{8}$$

The interaction of the quantum field is represented by the vertex factors

$$\begin{aligned} \mathcal{M}_3 &= -\epsilon^{abc}\left(D_\mu^{ad}Q_\nu^d\right)Q_\mu^b Q_\nu^c, \\ \mathcal{M}_4 &= -\frac{1}{4}Q_\mu^a Q_\nu^a Q_\mu^b Q_\nu^b + \frac{1}{4}Q_\mu^a Q_\nu^b Q_\mu^a Q_\nu^b. \end{aligned} \tag{9}$$

The complete Lagrangean

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}^a[B+Q]\right)^2 + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} \tag{10}$$

consists of (7), the gauge fixing term (in the following we put $\xi = 1$),

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}\left(D_\mu^a[B]Q_\mu^a\right)^2 = \frac{1}{2\xi}Q_\mu^a D_\mu^{ac}[B]D_\nu^{cb}[B]Q_\nu^b, \tag{11}$$

and the ghost term

$$\mathcal{L}_{\mathrm{gh}} = \overline{\eta}^a D_\mu^{ac}[B]\left(D_\mu^{cb}[B] + \epsilon^{cdb}Q_\mu^d\right)\eta^b. \tag{12}$$

These formulas are valid for an arbitrary background field. Now we turn to the specific background of an Abelian homogeneous magnetic field of strength B which is oriented along the third axis in both, color and configuration space. An explicit representation of its vector potential is

$$B_\mu^a(x) = \delta^{a3}\delta_{\mu 1}x_2\ B \tag{13}$$

and the corresponding field strength is

$$F_{ij}^a = \delta^{a3}F_{ij} = B\epsilon^{3ij}, \tag{14}$$

where only the spatial components $(i, j = 1, 2, 3)$ are non-zero. Once the background is chosen Abelian it is useful to turn to the so called charged basis,

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}\left(Q_\mu^1 \pm \mathrm{i}Q_\mu^2\right), \\ Q_\mu &= Q_\mu^3, \end{aligned} \tag{15}$$

with the interpretation of $W_\mu^\pm$ as color charged fields and Q_μ as color neutral field. This is in parallel to electrically charged and neutral fields. Note also that Q_μ is real while $W_\mu^\pm$ are complex conjugated one to the other. In the following we will omit the word color when speaking about charged and neutral objects. The same transformation is done for the ghosts,

$$\eta_\mu^\pm = \frac{1}{\sqrt{2}}\left(\eta_\mu^1 \pm \mathrm{i}\eta_\mu^2\right), \qquad \eta_\mu = \eta_\mu^3. \tag{16}$$

A summation over the color indices turns into

$$Q_\mu^a Q_\nu^a = Q_\mu Q_\nu + W_\mu^+ W_\nu^- + W_\mu^- W_\nu^+. \tag{17}$$

All appearing quantities have to be transformed into that basis. For the covariant derivative we get

$$D_\mu^{33} = \partial_\mu, \quad D_\mu^{-+} = \partial_\mu - \mathrm{i}B_\mu \equiv D_\mu$$
$$D_\mu^{+-} = \partial_\mu + \mathrm{i}B_\mu \equiv D_\mu^*, \tag{18}$$

where D_μ^* is the complex conjugated to D_μ. Starting from here we do not need any longer to indicate the arguments in the covariant derivatives.

Before proceeding with writing down the remaining formulas in the charged basis it is useful to turn to momentum representation. This can be done in a standard way by the formal rules. It remains to define the signs in the exponential factors. We adopt the notation

$$Q \sim \mathrm{e}^{-\mathrm{i}kx}, \qquad W^- \sim \mathrm{e}^{-\mathrm{i}px}, \qquad W^+ \sim \mathrm{e}^{\mathrm{i}p'x}. \tag{19}$$

In all following calculations the momentum k will denote the momentum of a neutral line and the momenta p and p' that of the charged lines whereby k and p are incoming and p' is outgoing. In this notation the covariant derivative D_μ acts on a W_μ^- and turns into

$$D_\mu = -\mathrm{i}(\mathrm{i}\partial_\mu + B_\mu) \equiv -\mathrm{i}p_\mu. \tag{20}$$

Note that the components of the momentum p_μ do not commute,

$$[p_\mu, p_\nu] = \mathrm{i}BF_{\mu\nu}. \tag{21}$$

In this notation the quadratic term of the action turns into

$$-\frac{1}{2}Q_\mu^a K_{\mu\nu}^{ab} Q_\nu^b \tag{22}$$
$$= \frac{1}{2}Q_\mu K_{\mu\nu}^{33} Q_\nu + \frac{1}{2}W_\mu^+ K_{\mu\nu}^{-+} W_\nu^- + \frac{1}{2}W_\mu^- K_{\mu\nu}^{+-} W_\nu^+,$$

with

$$K_{\mu\nu}^{33} \equiv K_{\mu\nu}(k) = \delta_{\mu\nu}k^2 - k_\mu k_\nu \tag{23}$$

and

$$K_{\mu\nu}^{-+} \equiv K_{\mu\nu}(p) = \delta_{\mu\nu}p^2 - p_\mu p_\nu + 2\mathrm{i}BF_{\mu\nu}. \tag{24}$$

We use the arguments k and p instead of the indices to indicate to which line a $K_{\mu\nu}$ belongs. The third term in the RHS of (22) is the same as the second one due to the complex conjugation rules. In the Feynman rules $(-K_{\mu\nu}^{33})^{-1}$ is the line for neutral gluons and is denoted by a wavy line and $(-K_{\mu\nu}^{-+})^{-1}$ is the line for charged gluons and is denoted by a directed solid line. We remark that these lines represent propagators in the background of the magnetic field. Frequently they are denoted by thick or double lines. Because we have in this paper no other lines the notation with ordinary (thin) lines is unique.

For later use we introduce here the set of eigenstates for the operators (23) and (24). For the color neutral states we can take exactly the same polarizations $\mid k, s\rangle$ as known from electrodynamics,

$$\mid k,1\rangle_\mu = \frac{1}{k_{||}}\begin{pmatrix}-k_2\\ k_1\\ 0\\ 0\end{pmatrix}_\mu,$$
$$\mid k,2\rangle_\mu = \frac{1}{kk_{||}}\begin{pmatrix}k_1k_3\\ k_2k_3\\ -k_{||}^2\\ 0\end{pmatrix}_\mu,$$
$$\mid k,3\rangle_\mu = \frac{1}{k}\begin{pmatrix}k_1\\ k_2\\ k_3\\ 0\end{pmatrix}_\mu, \quad \mid k,4\rangle_\mu = \begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix}_\mu, \tag{25}$$

with $k_{||} = \sqrt{k_1^2 + k_2^2}$, $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$. Here the polarizations $s = 1, 2$ describe the two transversal gluons ($k_\mu \mid k, s = 1, 2\rangle_\mu = 0$), $s = 3$ is the longitudinal one and $s = 4$ after rotation into Minkowski space becomes the time-like one. For the transversal gluons

$$K_{\mu\nu}(k) \mid k, s = 1, 2\rangle_\nu = (k_4^2 + k^2) \mid k, s = 1, 2\rangle_\mu \tag{26}$$

holds.

For the color charged gluons we denote the states by $\mid p_{||}, n, s\rangle_\mu$ where the integer n corresponds to the Landau levels and $p_{||} = \{p_3, p_4\}$ are the momenta in the direction parallel to the magnetic field. In opposite to the color neutral gluons whose momenta are completely continuous and whose wave functions are plane waves, for the color charged gluon we have a spectrum similar to an electrically charged particle in a magnetic field, continuous momenta in the directions parallel to the magnetic field (in our case $\mu = 3$ and $\mu = 4$) and the integer n which corresponds to the Landau levels. In fact there is one more quantum number which for example in the representation (13) describes the x-coordinate of the centers of the cyclotron orbits. We drop it because nothing we are interested in is in dependence on it.

In order to describe the states $\mid p_{||}, n, s\rangle_\mu$ it is useful to turn the coordinate system in the plane perpendicular to the magnetic field, i.e., in the (p_1, p_2)-plane, according to $p_\mu = B_{\mu\alpha}\, p_\alpha$ with

$$B_{\mu\alpha} = \begin{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \mathrm{i} & -i\end{pmatrix} & 0\\ 0 & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{pmatrix}_{\mu\alpha}, \tag{27}$$

where μ,ν are the Lorentz indices before and α,β after the rotation. The rotated momentum is

$$p_\alpha = \begin{pmatrix} \mathrm{i}a^\dagger \\ -\mathrm{i}a \\ p_3 \\ p_4 \end{pmatrix}_\alpha , \tag{28}$$

where $a = \frac{1}{\sqrt{2}}(\mathrm{i}p_1 - p_2)$ and $a^\dagger = \frac{1}{\sqrt{2}}(-\mathrm{i}p_1 - p_2)$ are the harmonic oscillator ladder operators, $[a, a^\dagger] = 1$. The field strength is diagonal in this representation,

$$F_{\alpha\beta} = \mathrm{i} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\alpha\beta} . \tag{29}$$

Now we denote by $\mid n\rangle$ the standard harmonic oscillator eigenstates, $\mid n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n \mid 0\rangle$. Together with a second harmonic oscillator which corresponds to the quantum number which we dropped they describe the coordinate dependence of the states in the plane perpendicular to the magnetic field. A fairly complete representation of these states (and of the tree level gluon propagator) we found in [12].

In this notation the states of the charged gluons are

$$\mid p_{||}, n, 1\rangle_\alpha = \frac{1}{h\sqrt{n(n+1)}} \begin{pmatrix} a^\dagger n \\ a(n+1) \\ 0 \\ 0 \end{pmatrix}_\alpha \mid n\rangle,$$

$$\mid p_{||}, n, 2\rangle_\alpha = \frac{1}{hk} \begin{pmatrix} \mathrm{i}a^\dagger l_3 \\ -\mathrm{i}a l_3 \\ -h^2 \\ 0 \end{pmatrix}_\alpha \mid n\rangle,$$

$$\mid p_{||}, n, 3\rangle_\alpha = \frac{1}{k} \begin{pmatrix} \mathrm{i}a^\dagger \\ -\mathrm{i}a \\ l_3 \\ 0 \end{pmatrix}_\alpha \mid n\rangle,$$

$$\mid p_{||}, n, 4\rangle_\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_\alpha \mid n\rangle, \tag{30}$$

where we introduced the notation $l^2 = p_3^2 + p_4^2$, $h^2 = (2n+1)B$, $k = \sqrt{p_3^2 + h^2}$, and

$$K_{\alpha\beta}(p) \mid p_{||}, n, s\rangle_\beta = (l^2 + h^2) \mid p_{||}, n, s\rangle_\alpha \qquad (s = 1, 2) \tag{31}$$

holds.

The tachyonic mode is

$$\mid p_{||}, -1, 1\rangle_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid 0\rangle. \tag{32}$$

It has $h^2 = -B$. In order to get it from (30) one needs to rewrite

$$\mid p_{||}, n, 1\rangle_\alpha = \frac{1}{h} \begin{pmatrix} \sqrt{n} \mid n+1\rangle \\ \sqrt{n+1} \mid n-1\rangle \\ 0 \\ 0 \end{pmatrix}_\alpha ,$$

using $a^\dagger \mid n\rangle = \sqrt{n+1} \mid n+1\rangle$ etc. There is no state with $n = -1$ for $s = 2$. Also, there is no state with $n = 0$ for $s = 1$, but one for $n = 0$, $s = 2$,

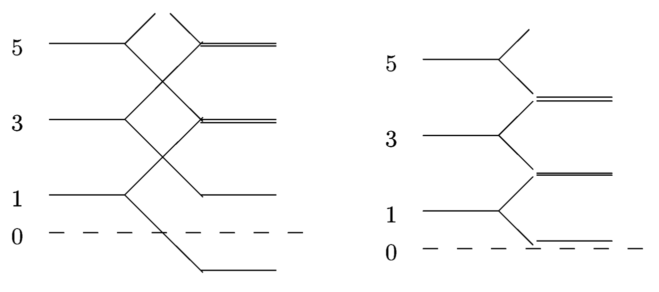
$$\mid p_{||}, 0, 2\rangle_\alpha = \frac{1}{k} \begin{pmatrix} \mathrm{i}p_3 \\ 0 \\ B \\ 0 \end{pmatrix}_\alpha \mid 0\rangle. \tag{33}$$

These are the two lowest states – they are singlets. All higher states have $n = 1, 2, \ldots$, $s = 1, 2$ and they are doublets. The states with $s = 1, 2$ are transversal, $p_\mu \mid p_{||}, n, s\rangle_\mu = 0$ $(s = 1, 2)$, i.e., they fulfill the subsidiary condition. The states (30), (32), (33) as well as (25) form each a basis of polarizations.

For completeness we note that after turning to Minkowski space the one particle energies $p_0^2 = -p_4^2$ of these states are $p_3^2 + h^2 = p_3^2 + B(2n+1)$ with $n = -1$ for the tachyonic mode (32). Further we note that for vanishing magnetic field these states turn into (25).

A more common representation would be $p_3^2 + B(2n+1\pm\sigma)$ with $n = 0, 1, \ldots$ for the Landau levels of a scalar field and with the spin projection $\sigma = \pm 1$ (for a spinor field $\sigma = \pm\frac{1}{2}$). A scheme of the levels in this representation is shown in Fig. 1 where for comparison the spinor case is shown too. To derive this representation in the literature the following discussion can be found. Taking into account the subsidiary condition, $p_\mu \mid n, s\rangle_\mu = 0$, (26) can be simplified by dropping $p_\mu p_\nu$ in (24). After that the remaining two operators, $\delta_{\mu\nu}p^2$ and $2\mathrm{i}F_{\mu\nu}$, commute and their common eigenvectors can be taken proportional to

$$\mid \sigma = 1\rangle \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mid \sigma = -1\rangle \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{34}$$



a $E_n^2 = B(2n+1\pm 2)$ **b** $E_n^2 = B\left(2n+1\pm 1\right)$

Fig. 1. The level scheme in a magnetic field, **a** for a color charged gluon, **b** for a spinor. Here n counts the Landau levels and it is different from the n in the states (30)

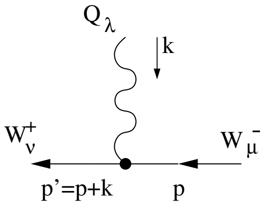


Fig. 2. Notation for the vertex $\mathcal{M}_3$

with the interpretation of spin up and spin down state. However, these two states cannot fulfill the subsidary condition. For that reason we prefer the states (30).

After discussing the free part of the Lagrangean (10), (7), in the "charged basis" we note that for the vertex factor $\mathcal{M}_3$ in (9) we get

$$\mathcal{M}_3 = W^-_\mu \Gamma_{\mu\nu\lambda} W^+_\nu Q_\lambda, \tag{35}$$

with

$$\Gamma_{\mu\nu\lambda} = g_{\mu\nu}(k-2p)_\lambda + g_{\lambda\mu}(p+k)_\nu + g_{\lambda\nu}(p-2k)_\mu. \tag{36}$$

The notation is shown in Fig. 2. Similar expressions appear for the vertices involving ghosts. It should be remarked that all graphs and combinatorical factors are exactly the same as in the well known case without magnetic field. On this level the only difference is in the meaning of the momentum p_μ which in our case depends on the background magnetic field; see (20).

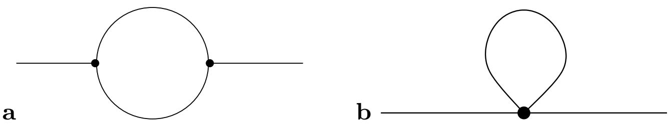


Fig. 3. a Basic graph for polarization tensor, **b** graph with one vertex and a closed line

The polarization tensor consists basically of the graphs shown in Fig. 3 where gluon and ghost lines must be inserted. For the calculation we use the following strategy. We drop all contributions which do not depend on an external momentum. This includes for instance the graph b in Fig. 3 with one vertex and a closed line over it. Also we will later integrate by parts in the parameter integrals and we will drop all boundary contributions. The justification for this procedure is that all these contributions in the final result must cancel. This had been observed explicitly in the calculation of the neutral polarization tensor in Fujikawa gauge. The general argument is that for homogeneity reasons the polarization tensor depends on the external momenta p only divided by the field strength, p^2/B. But for $B \to 0$ we have a normalization condition which removes all freedom in adding a constant. Another argument follows from the dispersion relations representing the polarization tensor in terms of its jump across the cut it has. But all contributions which we drop depend on the momentum only polynomially, hence they do not contribute to the jump.

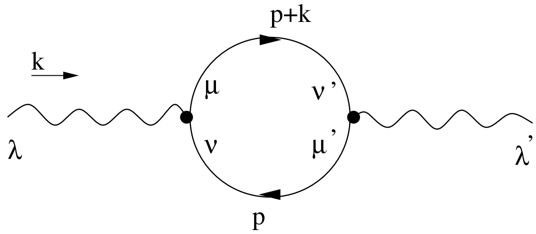


Fig. 4. The neutral polarization tensor

3 The neutral polarization tensor

The neutral polarization tensor is shown in Fig. 4. It is denoted by $\Pi_{\lambda\lambda'}(k)$ where the argument k indicates that it depends on an ordinary momentum which is a number in opposite to the momentum p. In the next subsection we discuss its general tensor structure and in the subsequent subsections we perform actually its calculation.

3.1 Operator structures

Since the polarization tensor is not transversal in a magnetic field, i.e., since $k_\lambda \Pi_{\lambda\lambda'}(k) = 0$ does not hold, we are left with the weaker condition

$$k_\lambda \Pi_{\lambda\lambda'}(k) k_{\lambda'} = 0. \tag{37}$$

It can be combined with the remaining Lorentz symmetry which results in a dependence of $\Pi_{\lambda\lambda'}(k)$ on two vectors, l_λ and h_λ, and on the magnetic field.

We use the notation

$$l_\mu = \begin{pmatrix} 0 \\ 0 \\ k_3 \\ k_4 \end{pmatrix}, \quad h_\mu = \begin{pmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{pmatrix}, \quad d_\mu = \begin{pmatrix} k_2 \\ -k_1 \\ 0 \\ 0 \end{pmatrix},$$

$$F_{\mu\lambda} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

The third vector is $d_\mu \equiv F_{\mu\nu} k_\nu$ with the field strength of the magnetic field which is connected with the non-commuting components of the momentum of a charged field by means of $[p_1, p_2] = \mathrm{i} F_{12}$. Here and in the rest of the section the magnetic field is put equal to unity. For the vectors $k_\lambda = l_\lambda + h_\lambda$ holds. We note that the components of the momentum k_μ are numbers which commute. The same holds in this section for h_μ.

The general structure of $\Pi_{\lambda\lambda'}(k)$ allowed by (37) and the vectors l_λ and h_λ is determined by the set of tensor structures

$$T^{(1)}_{\lambda\lambda'} = l^2 \delta^{||}_{\lambda\lambda'} - l_\lambda l_{\lambda'},$$
$$T^{(2)}_{\lambda\lambda'} = h^2 \delta^{\perp}_{\lambda\lambda'} - h_\lambda h_{\lambda'} = d_\lambda d_{\lambda'},$$
$$T^{(3)}_{\lambda\lambda'} = h^2 \delta^{||}_{\lambda\lambda'} + l^2 \delta^{\perp}_{\lambda\lambda'} - l_\lambda h_{\lambda'} - h_\lambda l_{\lambda'},$$

$$T^{(4)}_{\lambda\lambda'} = \mathrm{i}(l_\lambda d_{\lambda'} - d_\lambda l_{\lambda'}) + \mathrm{i}l^2 F_{\lambda\lambda'},$$
$$T^{(5)}_{\lambda\lambda'} = h^2 \delta^{||}_{\lambda\lambda'} - l^2 \delta^{\perp}_{\lambda\lambda'},$$
$$T^{(6)}_{\lambda\lambda'} = \mathrm{i}F_{\lambda\lambda'}, \tag{39}$$

together with the identity $d_\lambda h_{\lambda'} - h_\lambda d_{\lambda'} = h^2 F_{\lambda\lambda'}$. Further we introduced the notation $\delta^{\perp}_{\mu\lambda} = \mathrm{diag}(1,1,0,0)$ and $\delta^{||}_{\mu\lambda} = \mathrm{diag}(0,0,1,1)$. The first four operators are transversal, $k_\lambda T^{(i)}_{\lambda\lambda'} = T^{(i)}_{\lambda\lambda'} k_{\lambda'} = 0$ with $i = 1,2,3,4$, the last two fulfill (37) only. The sum of the first three operators is just the transversal part of the kernel of the quadratic part of the action, (23),

$$T^{(1)}_{\lambda\lambda'} + T^{(2)}_{\lambda\lambda'} + T^{(3)}_{\lambda\lambda'} = K_{\lambda\lambda'}(k). \tag{40}$$

Below we will be concerned with structures as they appear from the calculation of graphs. Here we collect some formulas which will be helpful to organize the results in terms of form factors. In general, knowing the operators (39) which may appear, the polarization tensor can be represented in the form

$$\Pi_{\lambda\lambda'}(k) = \sum_{i=1}^{6} \Pi^{(i)}(k) T^{(i)}_{\lambda\lambda'}, \tag{41}$$

where the "form factors", $\Pi^{(i)}(k)$, depend on l^2 and on h^2 only. Actually, since the polarization tensor $\Pi_{\lambda\lambda'}(k)$ is real and symmetric in its indices, the form factors $\Pi^{(4)}(k)$ and $\Pi^{(6)}(k)$ are zero.

In the course of the calculation, from the graph (Fig. 4) contributions will appear which have the following form. First,

$$\Pi_{\lambda\lambda'} = P_\lambda P^{\mathrm{T}}_{\lambda'} + a\delta^{||}_{\lambda\lambda'} + b\delta^{\perp}_{\lambda\lambda'} + \mathrm{i}cF_{\lambda\lambda'}, \tag{42}$$

where P_λ is given in terms of the vectors (38),

$$P_\lambda = rl_\lambda + \alpha \mathrm{i} d_\lambda + \beta h_\lambda. \tag{43}$$

The transposition in $P^{\mathrm{T}}_{\lambda'}$ changes the sign of $d_{\lambda'}$, $P^{\mathrm{T}}_{\lambda'} = rl_\lambda - \alpha \mathrm{i} d_\lambda + \beta h_\lambda$. The expression in (41) fulfills (37) if $\left(rl^2 + \beta h^2\right)^2 + al^2 + bh^2 = 0$ holds. In that case it can be represented in terms of form factors according to

$$\Pi_{\lambda\lambda'} = -r^2 T^{(1)}_{\lambda\lambda'} + (\alpha^2 - \beta^2) T^{(2)}_{\lambda\lambda'} - r\beta T^{(3)}_{\lambda\lambda'} - r\alpha T^{(4)}_{\lambda\lambda'} \tag{44}$$
$$+ \frac{r(rl^2 + \beta h^2) + a}{h^2} T^{(5)}_{\lambda\lambda'} + (ral^2 + \alpha\beta h^2 + c) T^{(6)}_{\lambda\lambda'}.$$

A second type of expressions appears which has a slightly more complicated form,

$$\Pi_{\lambda\lambda'} = P_\lambda Q_{\lambda'} + Q^{\mathrm{T}}_\lambda P^{\mathrm{T}}_{\lambda'} + a\delta^{||}_{\lambda\lambda'} + b\delta^{\perp}_{\lambda\lambda'} + \mathrm{i}cF_{\lambda\lambda'} \tag{45}$$

with P_λ from (43) and

$$Q_\lambda = sl_\lambda + \gamma \mathrm{i} d_\lambda + \delta h_\lambda. \tag{46}$$

If for (45) the condition (37) is fulfilled $(a + 2rsl^2)l^2 + (b + 2\beta\delta h^2)h^2 + (r\delta + s\beta)2l^2h^2 = 0$ must hold. In that case the representation in terms of form factors is

$$\Pi_{\lambda\lambda'} = -2rsT^{(1)}_{\lambda\lambda'} - 2(\beta\delta + \alpha\gamma)T^{(2)}_{\lambda\lambda'} - (r\delta + s\beta)T^{(3)}_{\lambda\lambda'}$$
$$+(r\gamma - s\alpha)T^{(4)}_{\lambda\lambda'} + \left((a + 2rsl^2)\frac{1}{h^2} + r\delta + s\beta\right) T^{(5)}_{\lambda\lambda'}$$
$$+ \left(c - (r\gamma - s\alpha)l^2 + (\alpha\delta - \beta\gamma)h^2\right) T^{(6)}_{\lambda\lambda'}. \tag{47}$$

3.2 Calculation of the neutral polarization tensor

The neutral polarization tensor has the following representation in momentum space (see Fig. 4):

$$\Pi_{\lambda\lambda'}(k) = \Gamma_{\mu\nu\lambda} G_{\mu\mu'}(p) \Gamma_{\mu'\nu'\lambda'} G_{\nu'\nu}(p-k)$$
$$-p_\lambda G(p)(p-k)_{\lambda'} G(p-k)$$
$$-(p-k)_\lambda G(p) p_{\lambda'} G(p-k), \tag{48}$$

where the integration over the momentum p is assumed. The second line is the contribution from the ghost loop.

The vertex factor is

$$\Gamma_{\mu\nu\lambda} = g_{\mu\nu}(k-2p)_\lambda + g_{\lambda\mu}(p+k)_\nu + g_{\lambda\nu}(p-2k)_\mu. \tag{49}$$

For a convenient grouping of terms it is useful to rearrange it,

$$\Gamma_{\mu\nu\lambda} = \underbrace{g_{\mu\nu}(k-2p)_\lambda} + \underbrace{2\left(g_{\lambda\mu}k_\nu - g_{\lambda\nu}k_\mu\right)}$$
$$+ \underbrace{g_{\lambda\mu}(p-k)_\nu + g_{\lambda\nu}p_\mu}$$
$$\equiv \Gamma^{(1)}_{\mu\nu\lambda} + \Gamma^{(2)}_{\mu\nu\lambda} + \Gamma^{(3)}_{\mu\nu\lambda}, \tag{50}$$

where in the last lines a subdivision into three parts is done.

The propagators are given by

$$G(p) = \frac{1}{p^2} = \int_0^\infty \mathrm{d}s \mathrm{e}^{-sp^2},$$
$$G(p-k) = \frac{1}{(p-k)^2} = \int_0^\infty \mathrm{d}t \mathrm{e}^{-t(p-k)^2} \tag{51}$$

for the scalar lines and by

$$G_{\lambda\lambda'}(p) = \left(\frac{1}{p^2 + 2\mathrm{i}F}\right)_{\lambda\lambda'} = \int_0^\infty \mathrm{d}s \mathrm{e}^{-sp^2} E^{-s}_{\lambda\lambda'},$$
$$G_{\lambda\lambda'}(p-k) = \left(\frac{1}{(p-k)^2 + 2\mathrm{i}F}\right)_{\lambda\lambda'} \tag{52}$$
$$= \int_0^\infty \mathrm{d}t \mathrm{e}^{-t(p-k)^2} E^{-t}_{\lambda\lambda'}$$

for the vector lines (in the Feynman gauge, $\xi = 1$) with

$$E^{s}_{\lambda\lambda'} \equiv \left(\mathrm{e}^{2\mathrm{i}sF}\right)_{\lambda\lambda'}$$
$$= \delta^{||}_{\lambda\lambda'} - \mathrm{i}F_{\lambda\lambda'} \sinh(2s) + \delta^{\perp}_{\lambda\lambda'} \cosh(2s). \tag{53}$$

The momentum integration can be carried out by means of Schwinger's known algebraic procedure [13] and converted into an integration over two scalar parameters, s and t. The basic exponential is

$$\Theta = \mathrm{e}^{-sp^2}\mathrm{e}^{-t(p-k)^2}, \quad (54)$$

and the integration over the momentum p is denoted by the average $\langle\ldots\rangle$. The following well known formulas hold:

$$\langle\Theta\rangle = \frac{\exp\left[-k\left(\frac{st}{s+t}\delta^{||}+\frac{ST}{S+T}\delta^{\perp}\right)k\right]}{(4\pi)^2(s+t)\sinh(s+t)}, \quad (55)$$

with $S=\tanh(s)$ and $T=\tanh(t)$ and

$$\langle p_\mu\Theta\rangle = \left(\frac{A}{D}k\right)_\mu \langle\Theta\rangle, \quad (56)$$

$$\langle p_\mu p_\nu\Theta\rangle = \left(\left(\frac{A}{D}k\right)_\mu\left(\frac{A}{D}k\right)_\nu - \mathrm{i}\left(\frac{F}{D^{\mathrm{T}}}\right)_{\mu\nu}\right)\langle\Theta\rangle. \quad (57)$$

The notation $A\equiv E^t-1$ and $D\equiv E^{s+t}-1$ is used. Explicit formulas are

$$\frac{A}{D} = \delta^{||}\frac{t}{s+t} - \mathrm{i}F\frac{\sinh(s)\sinh(t)}{\sinh(s+t)} + \delta^{\perp}\frac{\cosh(s)\sinh(t)}{\sinh(s+t)} \quad (58)$$

along with

$$\frac{-2\mathrm{i}FE^{-s}}{D^{\mathrm{T}}} = \frac{\delta_{||}}{s+t} - \mathrm{i}F\frac{\sinh(s-t)}{\sinh(s+t)} + \delta_{\perp}\frac{\cosh(s-t)}{\sinh(s+t)}, \quad (59)$$

where we dropped the indices. It should be remarked that all these matrices, i.e., E, F, D, commute. In addition we need the relation

$$p(s)_\mu \equiv \mathrm{e}^{-sp^2}p_\mu\mathrm{e}^{sp^2} = E^s{}_{\mu\nu}p_\nu$$

for commuting a factor p_μ with the propagator $G(p)$.

Within this formalism, the polarization tensor becomes an expression of the type

$$\Pi_{\lambda\lambda'}(k) = \int_0^\infty\int_0^\infty \mathrm{d}s\mathrm{d}t\langle M_{\lambda\lambda'}(p,k)\Theta\rangle, \quad (60)$$

where in $M_{\lambda\lambda'}(p,k)$ we collected all factors appearing from the vertices and from the lines except for that which go into Θ. By using (56) and (57) the average, i.e., the momentum integration over p, can be transformed into

$$\langle M_{\lambda\lambda'}(p,k)\Theta\rangle = M_{\lambda\lambda'}(s,t)\langle\Theta\rangle, \quad (61)$$

where now $M_{\lambda\lambda'}(s,t)$ collects all factors except for $\langle\Theta\rangle$.

To make these transformations manageable we break the whole polarization tensor into parts according to the division introduced in (50). We get

$$\Pi_{\lambda\lambda'}(k) = \sum_{i,j}\Pi^{ij}_{\lambda\lambda'}(k) + \Pi^{\mathrm{ghost}}_{\lambda\lambda'}(k), \quad (62)$$

with

$$\Pi^{ij}_{\lambda\lambda'}(k) = \Gamma^{(i)}_{\mu\nu\lambda}G_{\mu\mu'}(p)\Gamma^{(j)}_{\mu'\nu'\lambda}G_{\nu'\nu}(p-k) \quad (63)$$

and corresponding subdivisions of M.

Now we calculate the individual contributions. The first is

$$\Pi^{11}_{\lambda\lambda'} = (k-2p)_\lambda G_{\mu\mu'}(p)(k-2p)_{\lambda'}G_{\mu'\mu}(p-k) \quad (64)$$

and it transforms into

$$\begin{aligned}M^{11}_{\lambda\lambda'}(s,t)\langle\Theta\rangle &= \left\langle(k-2p)_\lambda(k-2p)_{\lambda'}E^{-s}_{\mu\mu'}E^{-t}_{\mu'\mu}\Theta\right\rangle\\ &= \left[\left(\left(1-2\frac{A}{D}\right)k\right)_\lambda\left(\left(1-2E^s\frac{A}{D}\right)k\right)_{\lambda'}\right.\\ &\qquad\left.-4\mathrm{i}\left(\frac{E^{-s}F}{D^{\mathrm{T}}}\right)_{\lambda\lambda'}\right]\mathrm{tr}E^{-s-t}\langle\Theta\rangle.\end{aligned} \quad (65)$$

We note the property $E^s\frac{A}{D}=\left(\frac{A}{D}\right)^{\mathrm{T}}$. The trace is

$$\begin{aligned}\mathrm{tr}E^{-s-t} &= \mathrm{tr}\left(\delta^{||}+\mathrm{i}F\sinh(2(s+t))+\delta^{\perp}\cosh(2(s+t))\right)\\ &= 2\left(1+\cosh(2(s+t))\right).\end{aligned} \quad (66)$$

The second part in this expression can be integrated by parts. We represent

$$\begin{aligned}\frac{-2\mathrm{i}FE^{-s}}{D^{\mathrm{T}}} &= \frac{1}{2}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right)\\ &\times\left[\frac{s-t}{s+t}\delta^{||} - \mathrm{i}F\frac{\cosh(s-t)}{\sinh(s+t)} + \delta^{\perp}\frac{\sinh(s-t)}{\sinh(s+t)}\right].\end{aligned} \quad (67)$$

The derivatives with respect to s and t will be integrated by parts. One should note that expressions symmetric under an exchange of s and t are not differentiated. So we have only the derivative of the exponential in Θ,

$$\frac{1}{2}\left(\partial_s-\partial_t\right)\langle\Theta\rangle = \frac{-1}{2}B_1\langle\Theta\rangle, \quad (68)$$

with the notation

$$B_1 \equiv \frac{s-t}{s+t}l^2 + \frac{\sinh(s-t)}{\sinh(s+t)}h^2, \quad (69)$$

which will frequently appear in the following.

Using the notation of (42) we define

$$\begin{aligned}P_\lambda &= \left(\left(1-2\frac{A}{D}\right)k\right)_\lambda\\ &= \frac{s-t}{s+t}l_\lambda + 2\mathrm{i}d_\lambda\frac{\sinh(s)\sinh(t)}{\sinh(s+t)} + h_\lambda\frac{\sinh(s-t)}{\sinh(s+t)}\\ &\equiv r\lambda + \mathrm{i}\alpha d_\lambda + \beta h_\lambda.\end{aligned} \quad (70)$$

Using (67) and (68) for the integration by parts and (70) we represent

$$M^{11}_{\lambda\lambda'}(s,t) = \Big\{P_\lambda P^{\mathrm{T}}_{\lambda'}$$

$$-\left[\frac{s-t}{s+t}\delta_{||} - \mathrm{i}F\frac{\cosh(s-t)}{\sinh(s+t)} + \delta_{\perp}\frac{\sinh(s-t)}{\sinh(s+t)}\right]_{\lambda\lambda'} B_1\Bigg\}$$
$$\times 2\left(1+\cosh(2(s+t))\right) \quad (71)$$

The contribution with the field strength F disappears if taking into account the symmetry of the integrals over s and t under exchange of these two arguments. Applying formula (44) with $a = \frac{s-t}{s+t}B_1$ and $b = \frac{\sinh(s-t)}{\sinh(s+t)}B_1$ we get

$$M^{11}(s,t) = \Bigg\{-\left(\frac{s-t}{s+t}\right)^2 T^{(1)} \quad (72)$$
$$+\left[\left(\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\right)^2 - \left(\frac{\sinh(s-t)}{\sinh(s+t)}\right)^2\right] T^{(2)}$$
$$-\frac{s-t}{s+t}\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}T^{(3)}\Bigg\}$$
$$\times 2\left(1+\cosh(2(s+t))\right).$$

All other contributions canceled.

Next contribution is M^{22}. From (50) we have

$$\Pi^{22}_{\lambda\lambda'} = 4\left(g_{\lambda\mu}k_\nu - g_{\lambda\nu}k_\mu\right)G_{\mu\mu'}(p) \times \left(g_{\lambda'\mu'}k_{\nu'} - g_{\lambda'\nu'}k_{\mu'}\right)G_{\nu'\nu}(p-k) \quad (73)$$

and

$$M^{22}_{\lambda\lambda'}(s,t) = 4\left(E^{-s}_{\lambda\lambda'}(kE^{-t}k) + E^{t}_{\lambda\lambda'}(kE^{-s}k)\right. \quad (74)$$
$$\left. - \left(E^{-s}k\right)_\lambda\left(E^{-t}k\right)_{\lambda'} - \left(E^{t}k\right)_\lambda\left(E^{s}k\right)_{\lambda'}\right).$$

Integration by parts is not required here. We introduced the short notation

$$(kE^{-t}k) \equiv k_\lambda E^{-t}_{\lambda\lambda'}k_{\lambda'} = l^2 + h^2\cosh(2t). \quad (75)$$

Here we have an expression of the type of (45). The parameter are

$$r = 1, \quad \alpha = \sinh(2t), \quad \beta = \cosh(2t),$$
$$s = 1, \quad \gamma = \sinh(2s), \quad \delta = \cosh(2s) \quad (76)$$

and

$$a = -2l^2 - h^2(\cosh(2s) + \cosh(2t)),$$
$$b = -l^2(\cosh(2s) + \cosh(2t)) - 2h^2\cosh(2s)\cosh(2t).$$

The tranversality condition is fulfilled and from (47) we obtain

$$M^{22}(s,t) = 8T^{(1)} + 8\cosh(2(s+t))T^{(2)} + 4(\cosh(2s) + \cosh(2t))T^{(3)}. \quad (77)$$

Next we consider Π^{12} and Π^{21}. From (50) we get

$$\Pi^{12}_{\lambda\lambda'} + \Pi^{21}_{\lambda\lambda'} \quad (78)$$
$$= g_{\mu\nu}(k-2p)_\lambda G_{\mu\mu'}(p)2\left(g_{\lambda'\mu'}k_{\nu'} - g_{\lambda'\nu'}k_{\mu'}\right) \times G_{\nu'\nu}(p-k)$$
$$+2\left(g_{\lambda\mu}k_\nu - g_{\lambda\nu}k_\mu\right)G_{\mu\mu'}(p)g_{\mu'\nu}(k-2p)_{\lambda'} \times G_{\nu'\nu}(p-k),$$

and further

$$\left(M^{12}_{\lambda\lambda'} + M^{21}_{\lambda\lambda'}\right)\langle\Theta\rangle$$
$$= \Big\langle\Big\{2(k-2p)_\lambda E^{-s}_{\nu\lambda'}k_{\nu'}E^{-t}_{\nu'\nu} - 2(k-2p)_\lambda E^{-s}_{\nu\mu'}k_{\mu'}E^{-t}_{\lambda'\nu}$$
$$+2k_\nu E^{-s}_{\lambda\nu'}(k-2p(s))_{\lambda'}E^{-t}_{\nu'\nu}$$
$$-2k_\mu E^{-s}_{\mu\nu'}(k-2p(s))_{\lambda'}E^{-t}_{\nu'\lambda}\Big\}\,\Theta\Big\rangle$$
$$= \Big\langle 2\left\{(k-2p)_\lambda\left(\left(E^{s+t} - E^{-s-t}\right)k\right)_{\lambda'}\right.$$
$$\left. + \left(\left(E^{s+t} - E^{-s-t}\right)k\right)_\lambda(k-2p(s))_{\lambda'}\right\}\Theta\Big\rangle. \quad (79)$$

We use the averages (56) and obtain an expression of the form of (45) with

$$P = 1 - 2\frac{A}{D}$$
$$= \frac{s-t}{s+t}\delta^{||} + \mathrm{i}F\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)} + \delta^{\perp}\frac{\sinh(s-t)}{\sinh(s+t)},$$
$$Q = E^{s+t} - E^{-s-t} = 2\mathrm{i}F\sinh(2(s+t)), \quad (80)$$

and from (47) we find simply

$$M^{12}(s,t) + M^{21}(s,t) = -32\cosh(s+t)\sinh(s)\sinh(t)T^{(2)}. \quad (81)$$

Next we consider the contribution of Π^{33} together with the contribution from the ghosts, Π^{ghost}. We get

$$\Pi^{33} = \left(g_{\lambda\mu}(p-k)_\nu + g_{\lambda\nu}p_\mu\right)G_{\mu\mu'}(p) \times \left(g_{\lambda'\mu'}(p-k)_{\nu'} + g_{\lambda'\nu'}p_{\mu'}\right)G_{\nu'\nu}(p). \quad (82)$$

We use the property of the propagator

$$p_\mu G_{\mu\mu'}(p) = G(p)p_{\mu'} \quad (83)$$

and obtain after simple calculation

$$\Pi^{33}_{\lambda\lambda'} = G_{\lambda\lambda'}(p)(p-k)^2G(p-k)$$
$$+p^2G(p)G_{\lambda\lambda'}(p-k)$$
$$+p_\lambda G(p)(p-k)_{\lambda'}G(p-k)$$
$$+(p-k)_\lambda G(p)p_{\lambda'}G(p-k).$$

In the first lines in the RHS one line collapses into a point by means of, e.g., $p^2G(p) = 1$ and the corresponding graph becomes a tadpole which we do not consider here. The next line in the RHS is just equal to the contribution from the ghosts, the second and third line in (48), with opposite sign, and cancels. So we obtain simply

$$\Pi^{33} + \Pi^{\mathrm{ghost}} = 0. \quad (84)$$

Next we consider Π^{13} and Π^{31}. We start from

$$\begin{aligned}\Pi^{13}_{\lambda\lambda'} &= g_{\mu\nu}(k-2p)_\lambda G_{\mu\mu'}(p)\\ &\quad\times\left(g_{\lambda'\mu'}(p-k)_{\nu'}+g_{\lambda'\nu'}p_{\mu'}\right)G_{\nu'\nu}(p-k),\\ \Pi^{31}_{\lambda\lambda'} &= \left(g_{\lambda\mu}(p-k)_\nu+g_{\lambda\nu}p_\mu\right)\\ &\quad\times G_{\mu\mu'}(p)g_{\mu'\nu'}(k-2p)_{\lambda'}G_{\nu'\nu}(p-k)\end{aligned} \tag{85}$$

and arrive at

$$\begin{aligned}M^{13}_{\lambda\lambda'}(s,t) &= -\left\langle (k-2p)_\lambda \left(E^{\rm sy}(k-2p)+E^{\rm as}k\right)_{\lambda'}\Theta\right\rangle,\\ M^{31}_{\lambda\lambda'}(s,t) &= -\left\langle \left(E^{\rm sy}(k-2p)-E^{\rm as}k\right)_\lambda (k-2p)_{\lambda'}\Theta\right\rangle,\end{aligned}$$

where the notation

$$\begin{aligned}E^{\rm sy} &= \frac{1}{2}\left(E^{s+t}+E^{-s-t}\right)=\delta^{||}+\delta^\perp\cosh(2(s+t)),\\ E^{\rm as} &= \frac{1}{2}\left(E^{s+t}-E^{-s-t}\right)={\rm i}F\sinh(2(s+t))\end{aligned} \tag{86}$$

has been introduced.

The averages can be calculated and we obtain

$$\begin{aligned}M^{13}_{\lambda\lambda'}(s,t) &= \left\{-P_\lambda Q_{\lambda'}+4{\rm i}\left(\frac{E^{\rm sy}E^{-s}F}{D^{\rm T}}\right)_{\lambda\lambda'}\right\}\langle\Theta\rangle,\\ M^{31}_{\lambda\lambda'}(s,t) &= \left\{-Q^{\rm T}_\lambda P^{\rm T}_{\lambda'}+4{\rm i}\left(\frac{E^{\rm sy}E^{-s}F}{D^{\rm T}}\right)_{\lambda\lambda'}\right\}\langle\Theta\rangle,\end{aligned} \tag{87}$$

with

$$\begin{aligned}P_\lambda &= \left(1-2\frac{A}{D}\right)_\lambda \qquad (88)\\ &= \frac{s-t}{s+t}l_\lambda+{\rm i}d_\lambda\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}+h_\lambda\frac{\sinh(s-t)}{\sinh(s+t)},\\ Q_{\lambda'} &= \left(\left(E^{\rm as}+E^{\rm sy}\left(1-2\frac{A}{D}\right)^{\rm T}\right)k\right)_{\lambda'}\\ &= \frac{s-t}{s+t}l_{\lambda'}\\ &\quad+{\rm i}d_{\lambda'}\left(\sinh(2(s+t))-\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\right)\\ &\quad+h_{\lambda'}\frac{\sinh(s-t)}{\sinh(s+t)}\cosh(2(s+t)).\end{aligned} \tag{89}$$

The second contributions to the RHS in (87) must be integrated by parts. Using (67) and (68) we get

$$\begin{aligned}&M^{13}_{\lambda\lambda'}(s,t)+M^{31}_{\lambda\lambda'}(s,t)\\ &=\left\{-P_\lambda Q_{\lambda'}-Q^\top_\lambda P^\top_{\lambda'} \qquad (90)\right.\\ &\quad+2\left[\frac{s-t}{s+t}\delta^{||}_{\lambda\lambda'}\right.\\ &\quad\left.\left.+\frac{\sinh(s-t)}{\sinh(s+t)}\cosh(2(s+t))\delta^\perp_{\lambda\lambda'}\right]B_1\right\}.\end{aligned}$$

The contribution with $F_{\lambda\lambda'}$ has been dropped again for symmetry reasons. Now we apply (45) with

$$\begin{aligned}a &= -2\frac{s-t}{s+t}\left[\frac{s-t}{s+t}l^2+\frac{\sinh(s-t)}{\sinh(s+t)}h^2\right]\\ b &= -2\frac{\sinh(s-t)}{\sinh(s+t)}\cosh(2(s+t))\\ &\quad\times\left[\frac{s-t}{s+t}l^2+\frac{\sinh(s-t)}{\sinh(s+t)}h^2\right]\end{aligned} \tag{91}$$

and obtain

$$\begin{aligned}\Pi^{13}+\Pi^{31} &= 2\left(\frac{s-t}{s+t}\right)^2 T^{(1)}\\ &+2\left\{\left[\left(\frac{\sinh(s-t)}{\sinh(s+t)}\right)^2-\left(\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\right)^2\right]\right.\\ &\times\cosh(2(s+t))\\ &\left.+\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\sinh(2(s+t))\right\}T^{(2)} \qquad (92)\\ &+\frac{s-t}{s+t}\frac{\sinh(s-t)}{\sinh(s+t)}\left(1+\cosh(2(s+t))\right)T^{(3)}.\end{aligned}$$

Finally we need Π^{23} and Π^{32}. Proceeding in the same way as before we derive from (48) and (50)

$$\begin{aligned}\Pi^{23}_{\lambda\lambda'} &= 2(g_{\lambda\mu}k_\nu-g_{\lambda\nu}k_\mu)G_{\mu\mu'}(p)\\ &\quad\times\left(g_{\lambda'\mu'}(p-k)_{\nu'}+g_{\lambda'\nu'}p_{\mu'}\right)G_{\nu'\nu}(p-k),\\ \Pi^{32}_{\lambda\lambda'} &= 2(g_{\lambda\mu}(p-k)_\nu+g_{\lambda\nu}p_\mu)G_{\mu\mu'}(p)\\ &\quad\times\left(g_{\lambda'\mu'}k_{\nu'}-g_{\lambda'\nu'}k_{\mu'}\right)G_{\nu'\nu}(p-k),\end{aligned} \tag{93}$$

which gives

$$\begin{aligned}&M^{23}_{\lambda\lambda'}(s,t)\langle\Theta\rangle\\ &=\Big\langle 2\left\{-E^{-s}_{\lambda\lambda'}\left(kE^t(k-p(s))\right)-E^t_{\lambda\lambda'}\left(kE^{-s}p(s)\right)\right.\\ &\quad+\left(E^{-s}p(s)\right)_\lambda\left(E^{-t}k\right)_{\lambda'}\\ &\quad\left.+\left(E^t(k-p(s))\right)_\lambda\left(E^sk\right)_{\lambda'}\right\}\Theta\Big\rangle,\\ &M^{32}_{\lambda\lambda'}(s,t)\langle\Theta\rangle\\ &=\Big\langle 2\left\{-E^{-s}_{\lambda\lambda'}\left(kE^{-t}(k-p)\right)-E^t_{\lambda\lambda'}\left(pE^{-s}k\right)\right.\\ &\quad+\left(E^{-s}k\right)_\lambda\left(E^{-t}(k-p)\right)_{\lambda'} \qquad (94)\\ &\quad\left.+\left(E^tk\right)_\lambda\left(E^sp\right)_{\lambda'}\right\}\Theta\Big\rangle.\end{aligned}$$

Now the average (56) will be used and the expression rearranged according to

$$M^{23}_{\lambda\lambda'}+M^{32}_{\lambda\lambda'}=A+B, \tag{95}$$

with

$$A = -4E^t_{\lambda\lambda'}\left(k\frac{E^t-1}{D}k\right) + 2\left(E^t k\right)_\lambda \left(\left(\frac{E^t-1}{D}\right)^{\mathrm{T}} k\right)_{\lambda'}$$

$$+ 2\left(\frac{E^t-1}{D}k\right)_\lambda \left(E^{-t}k\right)_{\lambda'},$$

$$B = -4E^{-s}_{\lambda\lambda'}\left(k\frac{E^s-1}{D}k\right) + 2\left(E^{-s} k\right)_\lambda \left(\frac{E^s-1}{D} k\right)_{\lambda'}$$

$$+ 2\left(\left(\frac{E^s-1}{D}\right)^{\mathrm{T}}k\right)_\lambda \left(E^{s}k\right)_{\lambda'}.$$

A and B have the structure of (45). For A we get

$$r=1, \quad \alpha=\sinh(2t), \quad \beta=\cosh(2t)$$
$$s=\frac{t}{s+t}, \quad \gamma=-\frac{\sinh(s)\sinh(t)}{\sinh(s+t)},$$
$$\delta=\frac{\cosh(s)\sinh(t)}{\sinh(s+t)} \tag{96}$$

and

$$a=-2\left(\frac{t}{s+t}l^2+\frac{\cosh(s)\sinh(t)}{\sinh(s+t)}h^2\right), \tag{97}$$
$$b=-2\cosh(2t)\left(\frac{t}{s+t}l^2+\frac{\cosh(s)\sinh(t)}{\sinh(s+t)}h^2\right),$$
$$c=-2\sinh(2t)\left(\frac{t}{s+t}l^2+\frac{\cosh(s)\sinh(t)}{\sinh(s+t)}h^2\right)$$

and for B

$$r=1, \quad \alpha=-\sinh(2s), \quad \beta=\cosh(2s)$$
$$s=\frac{s}{s+t}, \quad \gamma=\frac{\sinh(s)\sinh(t)}{\sinh(s+t)},$$
$$\delta=\frac{\sinh(s)\cosh(t)}{\sinh(s+t)} \tag{98}$$

and

$$a=-2\left(\frac{s}{s+t}l^2+\frac{\sinh(s)\cosh(t)}{\sinh(s+t)}h^2\right),$$
$$b=-2\cosh(2s)\left(\frac{s}{s+t}l^2+\frac{\sinh(s)\cosh(t)}{\sinh(s+t)}h^2\right),$$
$$c=2\sinh(2s)\left(\frac{s}{s+t}l^2+\frac{\cosh(s)\sinh(t)}{\sinh(s+t)}h^2\right). \tag{99}$$

Putting these contributions together we obtain with (47)

$$\Pi^{23}+\Pi^{32}$$
$$=2\Big\{-2T^{(1)}$$
$$-2\left[(\cosh(2t)\cosh(s)\sinh(t)\right.$$
$$+\cosh(2s)\sinh(s)\cosh(t))/(\sinh(s+t))$$
$$\left.-2\cosh(s-t)\sinh(s)\sinh(t)\right]T^{(2)}$$
$$-\left[1+\frac{s\cosh(2s)+t\cosh(2t)}{s+t}\right]T^{(3)} \tag{100}$$
$$\times\left[-1+\frac{s\cosh(2s)+t\cosh(2t)}{s+t}\right]T^{(5)}\Big\}.$$

In this way the contributions of the individual parts are calculated. Putting all together we get from (72), (81), (92) and (100) the final expression which we represent in terms of form factors according to (41),

$$\Pi^{(i)}(k)=\int_0^\infty\int_0^\infty \mathrm{d}s\mathrm{d}t M^{(i)}(s,t)\langle\Theta\rangle \tag{101}$$

with

$$M^{(1)}(s,t)=4-2\left(\frac{s-t}{s+t}\right)^2,$$
$$M^{(2)}(s,t) \tag{102}$$
$$=\left[\left(\frac{\sinh(s-t)}{\sinh(s+t)}\right)^2-\left(\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\right)^2\right]$$
$$\times\cosh(2(s+t))$$
$$+32\cosh(2(s+t))$$
$$\times\left(1-\cosh(s-t)\left(\cosh(s-t)+\cosh(s)\cosh(t)\right)\right)$$
$$-24\cosh(s+t)\sinh(s)\sinh(t),$$
$$M^{(3)}(s,t)$$
$$=-2\frac{s-t}{s+t}$$
$$\times\left[\frac{\sinh(s-t)}{\sinh(s+t)}\frac{1+\cosh(2(s+t))}{2}\right.$$
$$\left.+\frac{\cosh(2s)-\cosh(2t)}{2}\right]$$
$$-2+3(\cosh(2s)+\cosh(2t)),$$
$$M^{(5)}(s,t)=-2\left(1-\frac{\cosh(2s)+\cosh(2t)}{2}\right), \tag{103}$$

where we made again use of the symmetry under $s\leftrightarrow t$.

For vanishing magnetic field $B\to 0$ which means to take the lowest order in s and t we obtain

$$\Pi_{\lambda\lambda'}(k)=\int_0^\infty\int_0^\infty \mathrm{d}s\mathrm{d}t 2\left(2-\left(\frac{s-t}{s+t}\right)^2\right)$$
$$\times\left(T^{(1)}_{\lambda\lambda'}+T^{(2)}_{\lambda\lambda'}+T^{(3)}_{\lambda\lambda'}\right)\langle\Theta\rangle, \tag{104}$$

where here we have simply

$$\langle\Theta\rangle=(s+t)^{-2}\exp\left(-\frac{st}{s+t}k^2\right)$$

instead of (55).

3.3 Neutral polarization tensor in Fujikawa gauge

With the derived in the preceding section formulas at hand it is easy to calculate the neutral polarization tensor in Fujikawa gauge (for a definition see [10,11]). In that gauge the third part of the vertex factor, $\Gamma^{(3)}_{\mu\nu\lambda}$, is absent and the ghost contribution looks different,

$$\Pi^{\text{ghostFJ}}_{\lambda\lambda'} = (k-2p)_\lambda G(p)(k-2p)_{\lambda'} G(p-k). \tag{105}$$

It differs from the contribution $\Pi^{11}_{\lambda\lambda'}$, (64), by the different lines only. While in (64) the lines belong to a vector particle, in (105) they belong to a scalar one. The vertex factors in (64) and in (105) are the same. So the whole difference is in the absence of the factor given by (66) and in the different sign the ghosts enter with. In this way we get

$$M^{\text{ghostFJ}} = -2\left\{-\left(\frac{s-t}{s+t}\right)^2 T^{(1)}\right. \tag{106}$$

$$+\left[\left(\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\right)^2 - \left(\frac{\sinh(s-t)}{\sinh(s+t)}\right)^2\right] T^{(2)}$$

$$\left. -\frac{s-t}{s+t}\,\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)} T^{(3)}\right\}. \tag{107}$$

The polarization tensor in Fujikawa gauge is

$$\Pi^{\text{Fjgauge}}_{\lambda\lambda'} = \Pi^{11}_{\lambda\lambda'} + \Pi^{22}_{\lambda\lambda'} + \Pi^{12}_{\lambda\lambda'} + \Pi^{21}_{\lambda\lambda'} + \Pi^{\text{ghostFJ}}_{\lambda\lambda'}. \tag{108}$$

Taking together (72) and (107) along with (77) and (81) and using a representation like (101) and (41) we obtain for the form factors

$$M^{(1)}_{\text{Fj}} = 8 - 2\left(\frac{s-t}{s+t}\right)^2 \cosh(2(s+t)), \tag{109}$$

$$M^{(2)}_{\text{Fj}} = 2\left(4 + \left(\frac{2\sinh(s)\sinh(t)}{\sinh(s+t)}\right)^2 - \left(\frac{\sinh(s-t)}{\sinh(s+t)}\right)^2\right) \times \cosh(2(s+t))$$
$$-32\cosh(s+t)\sinh(s)\sinh(t) + 8\cosh(2(s+t)),$$

$$M^{(3)}_{\text{Fj}} = -2\frac{s-t}{s+t}\,\frac{\sinh(s-t)}{\sinh(s+t)}\cosh(s(s+t)) + 4\left(\cosh(2s)+\cosh(2t)\right).$$

In the limit without magnetic field we get in this case

$$\Pi^{\text{Fj}}_{\lambda\lambda'}(k) = \int_0^\infty\int_0^\infty \mathrm{d}s\mathrm{d}t 2\left(4 - \left(\frac{s-t}{s+t}\right)^2\right) \times \left(T^{(1)}_{\lambda\lambda'} + T^{(2)}_{\lambda\lambda'} + T^{(3)}_{\lambda\lambda'}\right)\langle\Theta\rangle, \tag{110}$$

a result which is different from (104).

3.4 The non-transversality of the neutral polarization tensor

The non-transversality of the neutral polarization tensor can be established by direct calculation before performing the momentum integration. For this to do we start from (48) and consider $\Pi_{\lambda\lambda'}(k)k_{\lambda'}$. We use the property of the vertex factor

$$\Gamma_{\mu'\nu'\lambda'}k_{\lambda'} = K_{\mu'\nu'}(p-k) - K_{\mu'\nu'}(p) \tag{111}$$

with $K_{\mu\nu}(p)$ defined by (8). We get

$$\Pi_{\lambda\lambda'}(k)k_{\lambda'} \tag{112}$$
$$= \Gamma_{\mu\nu\lambda}G_{\mu\mu'}(p)\left[K_{\mu'\nu'}(p-k) - K_{\mu'\nu'}(p)\right]G_{\nu\nu'}(p-k)$$
$$-p_\mu \frac{1}{p^2}\,(p-k)k\frac{1}{(p-k)^2}$$
$$-(p-k)_\mu\frac{1}{p^2}\,pk\frac{1}{(p-k)^2}.$$

We proceed with two times using the obvious relation (see (52))

$$K_{\mu\lambda}(p)G_{\lambda\nu}(p) = \delta_{\mu\nu} - p_\mu\frac{1}{p^2}p_\nu. \tag{113}$$

There will be two contributions resulting from $\delta_{\mu\nu}$ in (113). In these one denominator disappears and we drop these contributions. This is for the same reason as discussed in the end of Sect. 2. In the ghost terms we use $(p-k)p = \frac{1}{2}(p^2-k^2-(p-k)^2)$ and $pk = \frac{1}{2}(p^2+k^2-(p-k)^2)$. Again, the terms where one denominator cancels drop out and we have to keep only what is proportional to k^2. We obtain

$$\Pi_{\lambda\lambda'}(k)k_{\lambda'} \tag{114}$$
$$= -\Gamma_{\mu\nu\lambda}G_{\mu\mu'}(p)(p-k)_{\mu'}\frac{1}{(p-k)^2}(p-k)_\nu$$
$$+\Gamma_{\mu\nu\lambda}p_\mu\frac{1}{p^2}p_{\nu'}G_{\nu'\nu}(p-k) + \frac{1}{2}k_\lambda k^2\frac{1}{p^2(p-k)^2}.$$

Next we use the cyclic property of the trace and move in the first term the factor $(p-k)_\nu$ to the left side. After that we use $\Gamma_{\mu\nu\lambda}p_\mu = K_{\nu\lambda}(p-k) - K_{\nu\lambda}(k)$ and get

$$\Pi_{\lambda\lambda'}(k)k_{\lambda'} \tag{115}$$
$$= \left[K_{\lambda\mu}(p) - K_{\lambda\mu}(k)\right]G_{\mu\mu'}(p)(p-k)_{\mu'}\frac{1}{(p-k)^2}$$
$$-\left[K_{\nu\lambda}(p-k) - K_{\nu\lambda}(k)\right]\frac{1}{p^2}p_{\nu'}G_{\nu'\nu}(p-k)$$
$$+\frac{1}{2}k_\lambda k^2\frac{1}{p^2(p-k)^2}. \tag{116}$$

We claim that this expression is equal to

$$\Pi_{\lambda\lambda'}(k)k_{\lambda'} = K_{\lambda\mu}(k)\Sigma_\mu, \tag{117}$$

with

$$\Sigma_\mu \tag{118}$$

$$\equiv \left[G_{\mu\mu'}(p)(p-k)_{\mu'}\frac{1}{(p-k)^2}-\frac{1}{p^2}p_{\nu'}G_{\nu'\mu}(p-k)\right].$$

To obtain this we used for the first and for the third terms in (116) the property (113) and after that $p(p-k)=\frac{1}{2}(p^2-k^2+(p-k)^2)$ from which we keep k^2 only. Then it is seen that what remains compensates the contribution resulting from the ghosts (last term in RHS of (116)).

Now it is obvious that $k_\lambda \Pi_{\lambda\lambda'}(k)k_{\lambda'}=0$ holds. The RHS of (118) can be simplified. We use

$$p_\mu G_{\mu\mu'}(p)=\frac{1}{p^2}p_{\mu'}, \tag{119}$$

which is essentially a special case of (83), and get

$$\Sigma_\mu = k_\mu\frac{1}{p^2(p-k)^2}-G_{\mu\mu'}(p)k_{\mu'}\frac{1}{(p-k)^2} -\frac{1}{p^2}k_{\nu'}G_{\nu'\mu}(p-k). \tag{120}$$

Now we turn to the representation with the parametric integrals introduced in Sect. 3.2, especially the formulas (51) and (52). We obtain

$$\Sigma_\mu = \int \mathrm{d}s\int \mathrm{d}t\left(k_\mu - E^{-s}_{\mu\mu'}k_{\mu'}-k_{\nu'}E^{-t}_{\nu'\lambda}\right)\langle\Theta\rangle = \left(\left(-E^{-s}-E^{t}+1\right)k\right)_\mu\langle\Theta\rangle, \tag{121}$$

with $\langle\Theta\rangle$ given by (55).

In order to multiply with $K_{\lambda\mu}(k)$ we note

$$K_{\lambda\mu}(k)l_\mu = T^{(5)}_{\lambda\lambda'}k_{\lambda'} \text{ and } K_{\lambda\mu}(k)h_\mu = -T^{(5)}_{\lambda\lambda'}k_{\lambda'}. \tag{122}$$

In this way we get finally

$$\Pi_{\lambda\lambda'}(k)k_{\lambda'} = \int \mathrm{d}s\int \mathrm{d}t(-2+\cosh(s)+\cosh(t))\langle\Theta\rangle T^{(5)}_{\lambda\lambda'}k_{\lambda'}. \tag{123}$$

This corresponds just to the contribution of $M^{(5)}(s,t)$ in (103).

4 The charged polarization tensor

The charged polarization tensor is shown in Fig. 5. It is denoted by $\Pi_{\lambda\lambda'}(p)$ where the argument p is a momentum defined in (20). The calculations in this section are to a large extend in parallel to that in the preceding one for the neutral polarization tensor, however different in the details. For a number of objects we use here the same notation as in that section, however they have a different meaning and they are valid in this section only. The magnetic field is equal to unity in this section too.

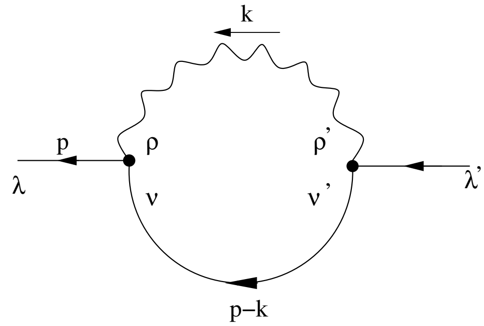


Fig. 5. The charged polarization tensor

4.1 Operator structures

The general structure of the charged polarization tensor is defined by its property not to be transversal but to obey the weaker condition

$$p_\lambda \Pi_{\lambda\lambda'}(p)p_{\lambda'}=0. \tag{124}$$

In this section we introduce the vectors l_μ and h_μ by

$$l_\mu=\begin{pmatrix}0\\0\\p_3\\p_4\end{pmatrix},\ h_\mu=\begin{pmatrix}p_1\\p_2\\0\\0\end{pmatrix},\ d_\mu=\begin{pmatrix}p_2\\-p_1\\0\\0\end{pmatrix},$$
$$F_{\mu\lambda}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \tag{125}$$

where the third vector is $d_\mu\equiv F_{\mu\nu}p_\nu$. This notation is valid in the present section only. Actually they are different from that in the preceding section by the non-commutativity of p, (21), only. We note $p_\lambda=l_\lambda+h_\lambda$ and the non-commutative piece is h_μ for which $[h_\mu,h_\nu]=\mathrm{i}F_{\mu\nu}$ holds. It must be remarked that the h_μ are operators. Later, when they act on the states like (30) the eigenvalues can be substituted according to $h^2\to 2n+1$. We make a distinction between the operator and its eigenvalues only where it is necessary for understanding. Furthermore it is useful to collect the following relations:

$$\begin{aligned} h_\lambda h_\lambda &= h^2, \quad d_\lambda d_\lambda = h^2,\\ [h_\lambda,h^2] &= 2\mathrm{i}d_\lambda, \quad [d_\lambda,h^2]=-2\mathrm{i}h_\lambda,\\ h_\lambda d_\lambda &= \mathrm{i}, \quad F_{\lambda\lambda'}d_{\lambda'}=-h_\lambda,\\ F_{\lambda\lambda'}h_{\lambda'} &= d_\lambda, \quad d_\lambda h_\lambda=-\mathrm{i},\\ d_\lambda F_{\lambda\lambda'} &= h_{\lambda'}, \quad h_\lambda F_{\lambda\lambda'}=-d_{\lambda'}. \end{aligned} \tag{126}$$

Now the general structure allowed by the remaining Lorentz invariance is

$$T^{(1)}_{\lambda\lambda'} = l^2\delta^{||}_{\lambda\lambda'}-l_\lambda l_{\lambda'},$$
$$T^{(2)}_{\lambda\lambda'} = h^2\delta^{\perp}_{\lambda\lambda'}+2\mathrm{i}F_{\lambda\lambda'}-h_\lambda h_{\lambda'} = d_\lambda d_{\lambda'}+\mathrm{i}F_{\lambda\lambda'},$$

$$
\begin{aligned}
T^{(3)}_{\lambda\lambda'} &= h^2\delta^{||}_{\lambda\lambda'} + l^2\delta^{\perp}_{\lambda\lambda'} - l_\lambda h_{\lambda'} - h_\lambda l_{\lambda'},\\
T^{(4)}_{\lambda\lambda'} &= \mathrm{i}(l_\lambda d_{\lambda'} - d_\lambda l_{\lambda'}) + \mathrm{i}l^2 F_{\lambda\lambda'} - \delta^{||}_{\lambda\lambda'},\\
T^{(5)}_{\lambda\lambda'} &= h^2\delta^{||}_{\lambda\lambda'} - l^2\delta^{\perp}_{\lambda\lambda'},\\
T^{(6)}_{\lambda\lambda'} &= \delta^{||}_{\lambda\lambda'} + 3\delta^{\perp}_{\lambda\lambda'} + (l^2+h^2)\mathrm{i}F_{\lambda\lambda'}
\end{aligned}
\tag{127}
$$

and the identity $\mathrm{i}(d_\lambda h_{\lambda'} - h_\lambda d_{\lambda'}) = \mathrm{i}h^2 F_{\lambda\lambda'} + \delta^{\perp}_{\lambda\lambda'}$ holds.

As in the preceding section the first four operators are transversal, $p_\lambda T^{(i)}_{\lambda\lambda'} = T^{(i)}_{\lambda\lambda'} p_{\lambda'} = 0$ for $i = 1,2,3,4$ and the remaining two fulfill (124) only, $p_\lambda T^{(i)}_{\lambda\lambda'} p_{\lambda'} = 0$ for $i = 5,6$. The sum of the first three operators is just the transversal part of the kernel of the quadratic part of the action, (24),

$$
T^{(1)}_{\lambda\lambda'} + T^{(2)}_{\lambda\lambda'} + T^{(3)}_{\lambda\lambda'} = K_{\lambda\lambda'}(p). \tag{128}
$$

The first three operators, $T^{(i)}_{\lambda\lambda'}$ with $i = 1,2,3$, commute. Hence they have common eigenvectors. These are some linear combinations of just the tree level states $\mid p_{||}, n, s\rangle_\alpha$, (30), of the charged gluons defined in Sect. 2. The following relations hold:

$$
\begin{aligned}
T^{(1)} \mid 1\rangle &= 0,\\
T^{(2)} \mid 1\rangle &= h^2 \mid 1\rangle,\\
T^{(3)} \mid 1\rangle &= l^2 \mid 1\rangle,\\
K \mid 1\rangle &= (l^2+h^2) \mid 1\rangle,\\
T^{(1)} \mid 2\rangle &= l_3 l_4 \frac{h}{k} \mid 0\rangle + \frac{l_4^2 h^2}{k^2} \mid 2\rangle - \frac{l_3 l_4^2 h}{k^2} \mid 3\rangle\\
T^{(2)} \mid 2\rangle &= 0,\\
T^{(3)} \mid 2\rangle &= -l_3 l_4 \frac{h}{k} \mid 0\rangle + \left(\frac{l_3^2 l_4^2}{k^2} + k^2\right) \mid 2\rangle + \frac{l_3 l_4^2}{k^2} \mid 3\rangle,\\
K \mid 2\rangle &= (l^2+h^2) \mid 2\rangle,
\end{aligned}
\tag{129}
$$

with $k = \sqrt{l_3^2 + h^2}$ and, because acting on the states, the operator h can be substituted by its eigenvalue, $h = \sqrt{2n+1}$. We dropped the Lorentz indices and some quantum numbers in the states. For example, the last term in the last line reads $K_{\alpha\beta}(p) \mid p_{||}, n, 2\rangle_\beta = (l^2+h^2) \mid p_{||}, n, 2\rangle_\alpha$ if the whole notation is kept. The tensors $T^{(i)}_{\lambda\lambda'}$ rotated according to (27) in the plane perpendicular to the magnetic field are shown in the appendix where also the matrix elements of the remaining tensors are listed.

From the calculation of the graph in the next subsection contributions will appear which have the same structure as given by the formulas (43), (45) and (46) in the preceding section and the coefficients in front of $T^{(1\ldots4)}$ in (44) are valid in this section too.

Before introducing the form factors a general remark is in order. The commutator relation (21) can be rewritten in the form $p_\lambda p^2 = (p^2\delta_{\lambda\lambda'} + 2\mathrm{i}F_{\lambda\lambda'})p_{\lambda'}$. As a consequence for any function of p^2 the relations

$$
p_\lambda f(p^2) = f(p^2 + 2\mathrm{i}F)_{\lambda\lambda'} p_{\lambda'} \tag{130}
$$

and

$$
f(p^2) p_\lambda = p_{\lambda'} f(p^2 + 2\mathrm{i}F)_{\lambda'\lambda} \tag{131}
$$

hold, where now f must be viewed as a function of a matrix so that it itself becomes a matrix carrying the indices λ and λ'. The same is true with h^2 in place of p^2.

Now the decomposition into form factors can be written as

$$
\Pi_{\lambda\lambda'}(p) = \sum_{i=1}^{6} \Pi^{(i)}(l^2, h^2 + 2\mathrm{i}F)_{\lambda\lambda''} T^{(i)}_{\lambda''\lambda'}. \tag{132}
$$

The property (130) ensures that (124) holds.

4.2 Strategy for the calculation of the charged polarization tensor

As compared with the neutral polarization tensor the calculation of the charged one is more difficult. A main complication is that it is unknown how to do the integration by parts which for the neutral tensor was done by the formulas (67) and (68). Fortunately there is a way out. First of all the contributions from the graph can be divided into such where the indices λ and λ' are attached to parts of the vector p, i.e. to l_λ, d_λ, h_λ and the corresponding ones with the index λ', and into such where the indices are attached to the constant tensors, i.e., to $\delta^{||}_{\lambda\lambda'}$, $\delta^{\perp}_{\lambda\lambda'}$ and $F_{\lambda\lambda'}$. Let us call these parts the $p_\lambda p_{\lambda'}$-part and the $\delta_{\lambda\lambda'}$-part respectively. Second, from the $p_\lambda p_{\lambda'}$-part the first four form factors, $\Pi^{(i)}$ with $i = 1,2,3,4$, can be restored uniquely. And third, from calculating $\Pi_{\lambda\lambda'}(p)p_{\lambda'}$, which can be done without the necessity to integrate by parts, the remaining two form factors can be restored uniquely.

For the restoration of the first four form factors from the $p_\lambda p_{\lambda'}$- part it is sufficient to consult the formulas (45) and (47). The $p_\lambda p_{\lambda'}$-part results from the quantities P_λ and Q_λ whereas the $\delta_{\lambda\lambda'}$-part results in the coefficients a, b and c in (45). Now from formula (47) it is seen that a, b and c do not enter the coefficients in front of the first four tensor structures, hence they do not enter the first four form factors.

For the restoration of the last two form factors we first calculate the non-transversal part, $\Pi_{\lambda\lambda'}(p)p_{\lambda'}$. The result will be obtained in the form

$$
\Pi_{\lambda\lambda'}(p)p_{\lambda'} = K_{\lambda\lambda'}(p)\Sigma_{\lambda'}(p). \tag{133}
$$

The fulfillment of (124) follows from the transversality of $K_{\lambda\lambda'}(p)$. The factor $\Sigma_{\lambda'}(p)$ has the form

$$
\Sigma_{\lambda'}(p) = \sigma_l l_\lambda + \sigma_d \mathrm{i}d_\lambda + \sigma_h h_\lambda, \tag{134}
$$

where the σ_i will be calculated explicitly; see Sect. 4.4. It remains to derive the easy formulas

$$
K_{\lambda\lambda'}(p) l_{\lambda'} = T^{(5)}_{\lambda\lambda'} p_{\lambda'}, \tag{135}
$$

$$
K_{\lambda\lambda'}(p) id_{\lambda'} = T^{(6)}_{\lambda\lambda'} p_{\lambda'}, \tag{136}
$$

$$
K_{\lambda\lambda'}(p) h_{\lambda'} = -T^{(5)}_{\lambda\lambda'} p_{\lambda'}, \tag{137}
$$

and we obtain for the RHS of (133)

$$K_{\lambda\lambda'}(p)\Sigma_{\lambda'}(p) = (\sigma_l - \sigma_h)T^{(5)}_{\lambda\lambda'}p_{\lambda'} + \sigma_d T^{(6)}_{\lambda\lambda'}p_{\lambda'}. \quad (138)$$

If we multiply (132) from the right by $p_{\lambda'}$ and compare with (133) we identify the last two form factors,

$$\begin{aligned} \Pi^{(5)}(p^2+2\mathrm{i}F) &= \sigma_l - \sigma_h, \\ \Pi^{(6)}(p^2+2\mathrm{i}F) &= \sigma_d. \end{aligned} \quad (139)$$

4.3 Calculation of the charged polarization tensor

The charged polarization tensor has the following representation in momentum space (see Fig. 5):

$$\begin{aligned} \Pi_{\lambda\lambda'}(p) &= \Gamma_{\lambda\nu\rho}G_{\nu\nu'}(p-k)\Gamma_{\lambda'\nu'\rho'}G_{\rho\rho'}(k) \\ &+(p-k)_\lambda G(p-k)k_{\lambda'}G(k) \\ &+k_\lambda G(p-k)(p-k)_{\lambda'}G(k), \end{aligned} \quad (140)$$

where the integration over the momentum k is assumed. The second line is the contribution from the ghosts. The vertex factor is the same as (49) but with renamed indices; it reads explicitly

$$\begin{aligned} \Gamma_{\lambda\nu\rho} &= (k-2p)_\rho g_{\lambda\nu} + g_{\rho\nu}(p-2k)_\lambda + g_{\rho\lambda}(p+k)_\nu, \\ &\equiv X_\rho g_{\lambda\nu} + g_{\rho\nu}Y_\lambda + g_{\rho\lambda}Z_\nu. \end{aligned} \quad (141)$$

In this section we use the parametric representation of the propagators in the following form:

$$G(p-k) = \int_0^\infty \mathrm{d}s\mathrm{e}^{-s(p-k)^2}, \quad G(k) = \int_0^\infty \mathrm{d}t\mathrm{e}^{-tk^2} \quad (142)$$

for the scalar lines and

$$\begin{aligned} G_{\nu\nu'}(p-k) &= \int_0^\infty \mathrm{d}s\mathrm{e}^{-s(p-k)^2}E_{\nu\nu'} \\ G_{\rho\rho'}(k) &= \int_0^\infty \mathrm{d}t\mathrm{e}^{-tk^2}g_{\rho\rho'} \end{aligned} \quad (143)$$

for the vector lines, with

$$E_{\nu\nu'} = \delta^{||}_{\lambda\lambda'} - \mathrm{i}F_{\lambda\lambda'}\sinh(2s) + \delta^{\perp}_{\lambda\lambda'}\cosh(2s), \quad (144)$$

which is the same as E^s in the preceding section. Since we use the Feynman gauge, $\xi = 1$, the propagator of the neutral gluon is $G_{\rho\rho'}(k) = g_{\rho\rho'}\ G(k)$.

The momentum integration which is over k here can be converted into the parametric integrals and averaging in an auxiliary space again following [13]. The basic exponential reads

$$\Theta = \mathrm{e}^{-s(p-k)^2}\mathrm{e}^{-tk^2}. \quad (145)$$

We denote the integration over the momentum k by the average $\langle\ldots\rangle$. It can be done, delivering

$$\langle\Theta\rangle(h^2) = \frac{\exp\left[-\frac{st}{s+t}l^2 - m(s,t)h^2\right]}{(4\pi)^2(s+t)\sqrt{N}}. \quad (146)$$

By the argument h^2 we indicated the dependence of the average on the external momentum. Of course it depends on l^2 too. Further, in (146) the notation

$$N = \frac{1}{4}\left((\sinh(2s)+2t)^2 - (\cosh(2s)-1)^2\right)$$

and

$$\begin{aligned} m(s,t) &\equiv \left(s - \operatorname{arctanh}\frac{\cosh(2s)-1}{\sinh(2s)+2t}\right) \\ &= \frac{1}{2}\ln\frac{1+2t-\mathrm{e}^{-2s}}{1-(1-2t)\mathrm{e}^{-2s}} \end{aligned} \quad (147)$$

are introduced. In view of (130) and the representation (132) in terms of form factors it is meaningful to rewrite (146) as

$$\begin{aligned} \langle\Theta\rangle(h^2) &= \frac{\exp\left[-\frac{st}{s+t}l^2 - m(s,t)(h^2+2\mathrm{i}F)\right]}{(4\pi)^2(s+t)\sqrt{N}}\mathcal{Z} \\ &\equiv \langle\Theta\rangle(p^2+2\mathrm{i}F)\mathcal{Z} \end{aligned} \quad (148)$$

with

$$\mathcal{Z} = -E^{\mathrm{T}}\frac{D}{D^{\mathrm{T}}}. \quad (149)$$

The quantities like D and A in the proceeding section appear here too, but they have a different meaning,

$$A = E - 1 \quad \text{and} \quad D = A - 2\mathrm{i}tF. \quad (150)$$

Explicitly we need the following combinations:

$$\begin{aligned} \frac{A}{-2\mathrm{i}F} &= s\delta^{||} - \frac{\mathrm{i}}{2}F(\cosh(2s)-1) + \frac{1}{2}\delta^{\perp}\sinh(2s), \\ \frac{A}{D} &= \frac{s}{s+t}\delta^{||} - \mathrm{i}F\frac{t(\cosh(2s)-1)}{2N} \\ &\quad +\delta^{\perp}\frac{\cosh(2s)-1+t\sinh(2s)}{2N}, \\ \frac{-2\mathrm{i}F}{D} &= \frac{1}{s+t}\delta^{||} + \mathrm{i}F\frac{\cosh(2s)-1}{2N} \\ &\quad +\delta^{\perp}\frac{\sinh(2s)+2t}{2N} \end{aligned} \quad (151)$$

and

$$\begin{aligned} \mathcal{Z}E &= -\frac{D}{D^{\mathrm{T}}} \\ &= \delta^{||} - \mathrm{i}F\frac{(\cosh(2s)-1)(\sinh(2s)+2t)}{2N} \\ &\quad +\delta^{\perp}\frac{(\cosh(2s)-1)^2+(\sinh(2s)+2t)^2}{4N}. \end{aligned} \quad (152)$$

We remind the reader that all these relations are to be understood as matrix multiplications and that all these matrixes commute. The matrix E has in addition the property $EE^{\mathrm{T}} = 1$ where E^{T} is the transposed to E. In the formulas (148) till (152) we dropped the indices λ and λ'.

The momentum p does not commute with Θ; instead

$$p_\lambda \Theta = \Theta p(s)_\lambda \tag{153}$$

holds with

$$p(s)_\lambda = (Ep)_\lambda - (Ak)_\lambda . \tag{154}$$

The averages of k_λ, i.e., the reductions of the momentum integration involving vectors to scalar integrations, read

$$\langle \Theta(p^2) k_\lambda \rangle = \langle \Theta(p^2) \rangle \left(\frac{A}{D} p \right)_\lambda \tag{155}$$

and

$$\langle \Theta(p^2) k_\lambda k_{\lambda'} \rangle = \langle \Theta(p^2) \rangle \left[\left(\frac{A}{D} p \right)_\lambda \left(\frac{A}{D} p \right)_{\lambda'} + \left(\frac{\mathrm{i}F}{D^{\mathrm{T}}} \right)_{\lambda\lambda'} \right] . \tag{156}$$

However, for the calculation of the $p_\lambda p_{\lambda'}$ terms we do not need the last term in the RHS. Next we use formula (148) and get the rules

$$\langle \Theta(p^2) k_\lambda \rangle \to \langle \Theta(p^2 + 2\mathrm{i}F) \rangle \left(\mathcal{Z} \frac{A}{D} p \right)_\lambda , \tag{157}$$

$$\langle \Theta(p^2) k_\lambda k_{\lambda'} \rangle \to \langle \Theta(p^2 + 2\mathrm{i}F) \rangle \left(\mathcal{Z} \frac{A}{D} p \right)_\lambda \left(\frac{A}{D} p \right)_{\lambda'} .$$

We apply these rules now to $\Pi_{\lambda\lambda'}(p)$, (140), and with formulas (141) to (143) we get

$$\Pi_{\lambda\lambda'}(p) = \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t \langle \Theta(p^2 + 2\mathrm{i}F) \rangle M(s,t)_{\lambda\lambda'}, \tag{158}$$

with

$$M(s,t)_{\lambda\lambda'} = \sum_{i=1}^{3} \left(P^{(i)}_\lambda Q^{(i)}_{\lambda'} + {Q^{(i)}}^{\mathrm{T}}_\lambda {P^{(i)}}^{\mathrm{T}}_{\lambda'} \right), \tag{159}$$

where we introduced the notation

$$\begin{aligned} P^{(1)}_\lambda &= \left(\mathcal{Z} \left(EX(s) + Y(s) \frac{\mathrm{tr}E}{2} + E^{\mathrm{T}} Z(s) \right) p \right)_\lambda , \\ Q^{(1)}_\lambda &= (Yp)_\lambda , \\ P^{(2)}_\lambda &= (\mathcal{Z} E Z p)_\lambda , \\ Q^{(2)}_\lambda &= (X(s) p)_\lambda , \\ P^{(3)}_\lambda &= (\mathcal{Z}(p(s) - k))_\lambda , \\ Q^{(3)}_\lambda &= k_\lambda . \end{aligned} \tag{160}$$

Here $X(s), Y(s)$ and $Z(s)$ are X, Y and Z with p substituted by $p(s)$, accordingly. We note that the relations $\mathcal{Z}X(s) = X^{\mathrm{T}}$, $\mathcal{Z}X = X^{\mathrm{T}}(s)$ hold. The same relations hold for Y and Z.

In (159) and (160) the individual contributions are grouped as follows. From (140) and (141) we get products involving two factors out from X, Y and Z in (141). Denoting these symbolically for the moment e.g. by XY, and dropping all other factors we have to consider all possible products. Some of them do not contribute to the $p_\lambda p_{\lambda'}$ terms, namely XX and ZZ. From the remaining ones we got $P^{(1)} = X + \frac{1}{2} Y + Z$ and $Q^{(1)} = Y$ so that the product $P^{(1)} Q^{(1)}$ in (159) covers the products XY, half of YY and ZY. The product ${Q^{(1)}}^{\mathrm{T}} {P^{(1)}}^{\mathrm{T}}$ in (159) covers YX, half of YY and YZ. In the same way we get $P^{(2)} = Z$ and $Q^{(2)} = X$ and conjugated. Finally, $P^{(3)}$ and $Q^{(3)}$ result from the ghost contributions. We note that in deriving these formulas in the contributions XZ and ZX we changed the order of the two p's appearing there. But their commutator results in a $\delta_{\lambda\lambda'}$ term.

Next we use formulas (151) and (152) to get explicit expressions for the P and Q in (160). After some calculation we get

$$\begin{aligned} P^{(1)} &= -\frac{s-t}{s+t} \cosh(2s) \delta^{||} + \left(3 \sinh(2s) - t \frac{p}{N} \right) \mathrm{i}F \\ &\quad + \left(-1 + t \frac{q}{N} \right) \delta^{\perp} , \\ Q^{(1)} &= -\frac{s-t}{s+t} \delta^{||} + t \frac{p}{N} \mathrm{i}F + \left(-1 + t \frac{q}{N} \right) \delta^{\perp} , \\ P^{(2)} &= \frac{2s+t}{s+t} \delta^{||} + \frac{p(t-q)}{N} \mathrm{i}F \\ &\quad + \left(\frac{p^2 + q^2}{2N} - \frac{tq}{2N} \right) \delta^{\perp} , \\ Q^{(2)} &= -\frac{s+2t}{s+t} \delta^{||} \\ &\quad + \left(\frac{\alpha}{4N} - t \frac{p \cosh(2s) - q \sinh(2s)}{2N} \right) \mathrm{i}F \\ &\quad + \left(-\frac{\beta}{4N} + t \frac{p \sinh(2s) - q \cosh(2s)}{2N} \right) \delta^{\perp} , \\ P^{(3)} &= \frac{t}{s+t} \delta^{||} - \frac{tp}{2N} \mathrm{i}F + t \frac{tq}{2N} \delta^{\perp} , \\ Q^{(3)} &= \frac{s}{s+t} \delta^{||} - \frac{tp}{2N} \mathrm{i}F + \left(1 - \frac{tq}{2N} \right) \delta^{\perp} , \end{aligned} \tag{161}$$

with

$$\begin{aligned} \alpha &= 4t(\cosh(2s) - 1) + 4t^2 \sinh(2s), \\ \beta &= 2(\cosh(2s) - 1) + 4t \sinh(2s) + 4t^2 \cosh(2s), \\ p &= \cosh(2s) - 1, \\ q &= \sinh(2s) + 2t. \end{aligned} \tag{162}$$

Now we remark that all expressions in (161) have just that structure which was described in (43) and (46) so that the corresponding r, s, α, β, γ and δ can be identified. Finally, we define a representation in parallel to (101) for the corresponding form factors,

$$\Pi^{(i)}(p) = \int_0^\infty \int_0^\infty \mathrm{d}s \mathrm{d}t \langle \Theta(p^2 + 2\mathrm{i}F) \rangle M^{(i)}(s,t) \tag{163}$$

and get

$$M^{(1)}(s,t) = 4 - 2\left(\frac{s-t}{s+t}\right)^2 \cosh(2s)$$

$$\begin{aligned} M^{(2)}(s,t) &= -2\left(-1+\frac{q}{N}\right)^2 - 2\left(3-\frac{tp}{N}\right)\frac{tp}{N} \\ &-2\left(\frac{p^2+q^2}{2N} - \frac{tq}{2N}\right) \\ &\times\left(\frac{-\beta}{4N} + t\frac{p\sinh(2s) - q\cosh(2s)}{2N}\right) \\ &-2\left(-\frac{pq}{N}+\frac{tp}{N}\right)\left(\frac{\alpha}{4N} - t\frac{p\cosh(2s)-q\sinh(2s)}{2N}\right) \\ &-\frac{tq}{N}\left(1-\frac{tq}{2N}\right) + \left(\frac{tp}{2N}\right)^2 \end{aligned}$$

$$\begin{aligned} M^{(3)}(s,t) &= \frac{s-t}{s+t}\left(-1+\frac{tq}{N}\right)(1+\cosh(2s)) \\ &-\frac{2s+t}{s+t}\left(\frac{-\beta}{2N} + t\frac{p\sinh(2s)-q\cosh(2s)}{2N}\right) \\ &+\frac{s+2t}{s+t}\left(\frac{p^2+q^2}{2N} - \frac{tq}{2N}\right) - \frac{t}{s+t}\left(1-\frac{tq}{2N}\right) \\ &-\frac{s}{s+t}\frac{tq}{2N} \end{aligned}$$

$$\begin{aligned} M^{(4)}(s,t) &= \frac{s-t}{s+t}\left(3-\frac{tp}{N}(1+\cosh(2s))\right) \\ &+\frac{2s+t}{s+t}\left(\frac{\alpha}{4N} - t\frac{p\cosh(2s)-q\sinh(2s)}{2N}\right) \\ &+\frac{s+2t}{s+t}\left(-\frac{pq}{N}+\frac{tp}{N}\right) + \frac{s-t}{s+t}\frac{tp}{2N}. \end{aligned} \tag{164}$$

In this way the first four form factors are found in a representation that only the parametric integrals remain to be done.

For vanishing magnetic field, which is equivalent to vanishing s and t, in the lowest order we obtain the same result as for the neutral polarisation tensor (cf. (104)),

$$M^{(i)} = 4 - 2\left(\frac{s-t}{s+t}\right)^2 \qquad (i=1,2,3), \tag{165}$$

and the $M^{(i)}$ with $i=4,5,6$ vanish (for $i=5,6$ see (177)).

4.4 The non-transversal part of the charged polarization tensor

Here we make use of some transformation under the sign of the momentum integration. We start from the obvoious formula

$$\Gamma_{\lambda'\nu'\rho'}p_{\lambda'} = K_{\nu'\rho'}(p-k) - K_{\nu'\rho'}(k). \tag{166}$$

Further we use (113) and (119). As before we drop all contributions in which one of the two denominators cancels against a factor in the numerator. In the ghost contribution we use the identity $kp = \frac{1}{2}(p^2+k^2-(p-k)^2)$. In this way we get in the next step

$$\begin{aligned} &\Pi_{\lambda\lambda'}(p)p_{\lambda'} \\ &= \Gamma_{\lambda\nu\rho}G_{\nu\nu'}(p-k)\left[K_{\nu'\rho}(p-k) - K_{\nu'\rho}(k)\right]\frac{1}{k^2} \\ &\quad +(p-k)_\lambda\frac{1}{(p-k)^2}pk\frac{1}{k^2} + k_\lambda\frac{1}{(p-k)^2}(p-k)k\frac{1}{k^2} \\ &= -\Gamma_{\lambda\nu\rho}(p-k)_\nu\frac{1}{(p-k)^2}(p-k)_\rho\frac{1}{k^2} \\ &\quad +\Gamma_{\lambda\nu\rho}k_\rho G_{\nu\nu'}(p-k)k_{\nu'}\frac{1}{k^2} + \frac{1}{2}p_\lambda\frac{1}{(p-k)^2}p^2\frac{1}{k^2}. \end{aligned} \tag{167}$$

Continuing with the mentioned manipulations we get

$$\begin{aligned} &\Pi_{\lambda\lambda'}(p)p_{\lambda'} \\ &= -\left[K_{\lambda\rho}(p) - K_{\lambda\rho}(k)\right]\frac{1}{(p-k)^2}(p-k)_\rho\frac{1}{k^2} \\ &\quad +\left[K_{\lambda\nu}(p-k) - K_{\lambda\nu}(p)\right]G_{\nu\nu'}(p-k)k_{\nu'}\frac{1}{k^2} \\ &\quad +\frac{1}{2}p_\lambda\frac{1}{(p-k)^2}p^2\frac{1}{k^2} - K_{\lambda\rho}(p)\frac{1}{(p-k)^2}(p-k)_\rho\frac{1}{k^2} \\ &\quad -\frac{1}{(p-k)^2}k_\lambda(p-k)k\frac{1}{k^2} \\ &\quad -(p-k)_\lambda\frac{1}{(p-k)^2}(p-k)k\frac{1}{k^2} \\ &\quad -K_{\lambda\nu}(p)G_{\nu\nu'}(p-k)k_{\nu'}\frac{1}{k^2} + \frac{1}{2}p_\lambda\frac{1}{(p-k)^2}p^2\frac{1}{k^2}. \end{aligned} \tag{168}$$

Now the ghost contribution cancels against the second and the third terms. Finally we obtain

$$\Pi_{\lambda\lambda'}(p)p_{\lambda'} = K_{\lambda\rho}(p)\Sigma_\rho(p), \tag{169}$$

with

$$\Sigma_\rho(p) = \left(\frac{-1}{(p-k)^2}(p-k)_\rho - G_{\rho\nu'}(p-k)k_{\nu'}\right)\frac{1}{k^2}. \tag{170}$$

From (169) it is obvious that $p_\lambda\Pi_{\lambda\lambda'}(p)p_{\lambda'} = 0$ holds. Now we calculate the RHS of (170). Using the same formulas as in Sect. 4.3 we get

$$\begin{aligned} \Sigma_\lambda(p) &= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t \\ &\times\left(\langle\Theta(p^2+2\mathrm{i}F)\rangle\mathcal{Z}\left(-(p-k)-Ek\right)\right)_\lambda. \end{aligned} \tag{171}$$

Using formulas (151) and (152) we transform

$$\mathcal{Z}\left(-(p-k)-Ek\right) = \mathcal{Z}\left(-1-\frac{A^2}{D}\right)p, \tag{172}$$

and with (150) we come to

$$\mathcal{Z}\left(-1-\frac{A^2}{D}\right) = \frac{E^{\mathrm{T}}}{D^{\mathrm{T}}}(A-2\mathrm{i}tF+A^2) \tag{173}$$
$$= -\left(\frac{A}{-2\mathrm{i}F}+tE^{\mathrm{T}}\right)\left(\frac{-2\mathrm{i}F}{D}\right)^{\mathrm{T}}.$$

Now we use the explicit expressions from (151) and obtain

$$\begin{aligned}
&\mathcal{Z}\left(-1-\frac{A^2}{D}\right)\\
&= -\delta^{||}\\
&\quad +\mathrm{i}F\frac{1}{4N}\,((\cosh(2s)-1)(\sinh(2s)+2t\cosh(2s))\\
&\quad -(\sinh(2s)+2t)(-\cosh(2s)+1+2t\sinh(2s)))\\
&\quad +\delta^{\perp}\frac{1}{4N}\\
&\quad \times((\cosh(2s)-1)(-\cosh(2s)+1+2t\sinh(2s))\\
&\quad -(\sinh(2s)+2t)(\sinh(2s)+2t\cosh(2s)))\,.
\end{aligned} \tag{174}$$

Further we multiply by the vector p and with (134) we identify

$$\begin{aligned}
\sigma_l &= \int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}t\langle\Theta(p^2+2\mathrm{i}F)\rangle\sigma_l(s,t),\\
\sigma_d &= \int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}t\langle\Theta(p^2+2\mathrm{i}F)\rangle\sigma_d(s,t),\\
\sigma_h &= \int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}t\langle\Theta(p^2+2\mathrm{i}F)\rangle\sigma_h(s,t),
\end{aligned} \tag{175}$$

with

$$\begin{aligned}
\sigma_l(s,t) &= 1,\\
\sigma_d(s,t) &= \frac{\sinh(2s)(2t^2-\cosh(2s)+1)}{2N},\\
\sigma_h(s,t) &= (\cosh(2s)(\cosh(2s)-1)\\
&\qquad +2t\sinh(2s)+2t^2\cosh(2s))\\
&\qquad /(2N),
\end{aligned} \tag{176}$$

where some simplifications had been made. Finally we get with (139) the last two form factors in the representation (163),

$$\begin{aligned}
M^{(5)}(s,t) &= 1\\
&\quad -(\cosh(2s)(\cosh(2s)-1)+2t\sinh(2s)\\
&\quad +2t^2\cosh(2s))\\
&\quad /(2N),\\
M^{(6)}(s,t) &= \frac{\sinh(2s)(2t^2-\cosh(2s)+1)}{2N}.
\end{aligned} \tag{177}$$

By these formulas the calculation of the polarization tensor of the charged gluons is finished.

5 Renormalization of the charged polarization tensor

In order to discuss the renormalization of the charged polarization tensor we change the notation and represent the form factors which are given by (158) and (163) as

$$\Pi^{(i)}(p) = \int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}t\frac{M^{(i)}(s,t)}{(s+t)\sqrt{N}}\ \mathrm{e}^{-H(s,t)}, \tag{178}$$

with

$$H(s,t) = \frac{st}{s+t}l^2+m(s,t)\ (h^2+2\mathrm{i}F), \tag{179}$$

where the $M^{(i)}(s,t)$ are given by (164) and (177) and $m(s,t)$ by (147). We change variables according to $s=\lambda x$, $t=\lambda(1-x)$ and rewrite (178) as

$$\Pi^{(i)}(p) = \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}f_{\mathrm{i}}(\lambda), \tag{180}$$

with

$$f_{\mathrm{i}}(\lambda) = \int_0^1 \mathrm{d}x\frac{\lambda M^{(i)}(s,t)}{\sqrt{N}}\mathrm{e}^{-H(s,t)}. \tag{181}$$

For convenience of representation we keep the variables s and t although they are to be substituted by λ and x.

Now we restore the dependence on the magnetic field B. This can be done by substituting $\lambda\to\lambda B$ and $l^2\to l^2/B$ in $f_{\mathrm{i}}(\lambda)$. We recall that $\Pi^{(i)}(p)$ is dimensionless. Further we use the representation in Minowski space which can be obtained from the Euclidean representation which we used so far by rotating "back", i.e., by substituting $\lambda\to -\mathrm{i}\lambda$ and adding a "$-\lambda\epsilon$" in the exponential defining the causal propagator in the standard way with $\epsilon>0$. The momentum becomes $l^2\to -l_0^2+l_3^2$. Further we introduce an ultraviolet regularization (with regularization parameter $\delta>0$ and arbitrary massive parameter μ) and come to the representation

$$\Pi^{(i)\mathrm{reg}}(p) = \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}(\lambda\mu^2)^\delta f_{\mathrm{i}}(-\mathrm{i}\lambda B)\mathrm{e}^{-\lambda\epsilon}. \tag{182}$$

The ultraviolet divergences come from the small λ. We note that in the limit of $\lambda\to 0$ the $f_{\mathrm{i}}(\lambda)$ for $i=1,2,3$ become just that for a polarization tensor without magnetic field and for $i=4,5,6$ they vanish (see (104) and (165)). Taking into account (128) in this way the renormalization can be done indeed by subtracting all contributions which do not depend on the magnetic field. By adding and subtracting $f_{\mathrm{i}}(0)$ they can be separated,

$$\Pi^{(i)\mathrm{div}}(p) = \left(\frac{1}{\delta}+\ln\mu^2\right)f_{\mathrm{i}}(0), \tag{183}$$

and the renormalized form factors are

$$\begin{aligned}
&\Pi^{(i)\mathrm{ren}}(p)\\
&= \ln\lambda_0 f_{\mathrm{i}}(0)+\int_0^{\lambda_0}\frac{\mathrm{d}\lambda}{\lambda}(f_{\mathrm{i}}(-\mathrm{i}\lambda B)-f_{\mathrm{i}}(0))
\end{aligned}$$

$$+\int_{\lambda_0}^{\infty}\frac{\mathrm{d}\lambda}{\lambda}f_{\mathrm{i}}(-\mathrm{i}\lambda B)\mathrm{e}^{-\lambda\epsilon}, \tag{184}$$

where we divided the integration region at an arbitrary point λ_0. Obviously (184) does not depend on λ_0.

In this way we obtained the renormalized form factors and, strictly speaking, the final expression. It is finite and can be calculated numerically. But it is useful to give an numerical example which, for instance, demonstrates the appearance of the imaginary part resulting from the tachyonic mode. The most typical example one can think of is to take the polarization tensor averaged in the unperturbed tree level states (30). This gives the first perturbative correction,

$$\Delta E_n^2=\langle p_{||},n,s\,|_{\alpha}\ \Pi_{\alpha\beta}(p)\ |\ p_{||},n,s\rangle_{\beta}, \tag{185}$$

to the tree level energy $E^2_{\mathrm{tree}}=p_3^2+B(2n+1)$ (cf. (31)). Taking into account the matrix elements (A3) in the physical states $s=1,2$ from representation (132) of the polarization tensor we get

$$\begin{aligned}&\Delta E_n^2(s=1)\\&=(\Pi^{(2)}-\Pi^{(3)}+\Pi^{(5)})(2n+1)-\Pi^{(4)}+\Pi^{(6)},\\&\Delta E_n^2(s=2)\\&=(-\Pi^{(1)}+\Pi^{(3)}+\Pi^{(5)})(2n+1)-\Pi^{(4)}+\Pi^{(6)},\end{aligned} \tag{186}$$

where the on-shell condition $-l_4^2=l_0^2=p_3^2+B(2n+1)$ was used.

It must be noticed, however, that the form factors $\Pi^{(i)}(p^2_{||},p^2_{\perp})$ have on-shell a logarithmic infrared singularity which is typical for a graph with massless particles taken on-shell. We regularize this divergence by an auxiliary gluon mass m_g and perform the calculation for $m_g\to 0$. This mass is, of course, the same as the parameter ϵ in (184) so that we substitute $\epsilon\to \mathrm{i}m_g^2$ there. Now the form factors depend only on the magnetic field, on the auxiliary mass and on the number n of the states in (199) and the calculation is reduced to a numerical evaluation of the double parametric integrals in (181) and (184). However, in this form the integrand is an oscillating function and not very friendly for numerical evaluation. It would be useful to perform the Wick rotation. It is known that this is impossible in a direct way because the euclidean integral diverges exponentially due to the tachyonic mode. We demonstrate a way to overcome this problem. We start from representation (184) taken on-shell with the auxiliary mass $\epsilon\to \mathrm{i}m_g^2$. After that we rescale the variable λ by $\lambda\to\lambda/B$ and make the choice $\lambda_0=1/B$. After that the magnetic field enters only through the momentum in the combination l^2/B and h^2/B in $H(s,t)$, (179). In Minkowski space the momentum is $l^2=-l_0^2+l_3^2$ and on-shell we have $l_0^2=l_3^2+B(2n+1)$ and $h^2=B(2n+1)$ so that with $l^2/B\to(2n+1)$ the B-dependence drops out, leaving behind only the dependence on the number n of the Landau level. Of course, now the first term in the RHS of (184) contains ln B.

The tachyonic mode manifests itself in the behavior of the functions $M^{(i)}(s,t)$ for large s. The following holds:

$$M^{(i)}(s,t)=\mathrm{e}^{2s}m_{\mathrm{i}}(s,t)+O\left(1\ \right) \tag{187}$$

for large s, and the factor e^{2s} determines the leading behavior of the whole integrand. The functions $m_{\mathrm{i}}(s,t)$ read

$$\begin{aligned}&m_1(s,t)=-\left(\frac{s-t}{s+t}\right),\quad m_2(s,t)=\frac{2(1+t)}{1+2t},\\&m_3(s,t)=\frac{s+5t}{2(s+t)(1+2t)},\\&m_4(s,t)=\frac{s+4st-t(7+4t)}{2(s+t)(1+2t)},\\&m_5(s,t)=\frac{1}{2(1+2t)},\quad m_6(s,t)=\frac{-1}{2(1+2t)}.\end{aligned} \tag{188}$$

We divide the whole expression for the form factor into what comes from (187) (part A) and the rest (part B),

$$\begin{aligned}f_{\mathrm{i}}^{\mathrm{A}}(\lambda)&=\int_0^1\mathrm{d}x\mathrm{e}^{2s}m_{\mathrm{i}}(s,t)\frac{\lambda}{\sqrt{N}}\ \mathrm{e}^{-H(s,t)},\\f_{\mathrm{i}}^{\mathrm{B}}(\lambda)&=\int_0^1\mathrm{d}x\left(M^{(i)}(s,t)-\mathrm{e}^{2s}m_{\mathrm{i}}(s,t)\ \right)\frac{\lambda}{\sqrt{N}}\mathrm{e}^{-H(s,t)}.\end{aligned} \tag{189}$$

Further we remark that on-shell the function in the exponential is

$$\begin{aligned}H(s,t)&=\left(-\frac{st}{s+t}+m(s,t)\right)(2n+1)\\&\equiv -h(s,t)(2n+1).\end{aligned} \tag{190}$$

The function $h(s,t)$ has the properties

$$\begin{aligned}&h(-s,-t)=-h(s,t),\\&0\le h(s,t)\le\frac{1}{4}\lambda\ \ (s,t\ge 0),\\&h(\lambda x,\lambda(1-x))=\frac{1}{3}\lambda^3x^4+O(x^5),\end{aligned} \tag{191}$$

from which it follows that for $n=-1$ the leading behavior of $f_{\mathrm{i}}^{\mathrm{A}}(\lambda)$ for $\lambda\to\infty$ is indeed determined by the factor e^{2s}. For higher n one had to include more orders in the expansion (187) so that we restrict ourselves to $n=-1$ in the following.

In this way the usual Wick rotation is impossible to do in part A. This is in line with the picture in momentum space. Using the causal propagator we have the usual poles below the positive real k_0-axis and above the negative one. In addition there are poles from the tachyonic mode which are located on the opposite side of the imaginary axis. In these contributions the Wick rotation would deliver additional contributions resulting from these poles. But it is possible to rotate the contour clockwise, i.e., in the opposite direction, $\lambda\to-\mathrm{i}\lambda$. We do this in part A. Of course, in part B we rotate in the usual direction, $\lambda\to\mathrm{i}\lambda$.

We are going to do these rotations in formula (184). However, there the integration region is divided into two parts with different functions. To get one integral we integrate by parts and obtain

$$\Pi^{(i)\mathrm{ren}}(p)=-\ln B f_{\mathrm{i}}(0) \tag{192}$$

$$-\int_0^\infty d\lambda \ln\lambda \frac{d}{d\lambda}\left(f_{\rm i}^{\rm A}(-{\rm i}\lambda)+f_i^{\rm B}(-{\rm i}\lambda)\right) {\rm e}^{{\rm i}\lambda m_g^2}.$$

The $f_{\rm i}(0)$ follow from (165) after integration over x, and their values are

$$f_{\rm i}(0)=\tfrac{10}{3} \text{ for } i=1,2,3 \quad\text{and}\quad f_{\rm i}(0)=0 \text{ for } i=4,5,6. \tag{193}$$

Now we perform the Wick rotation in part B and the opposite one in part A. From $\ln\lambda$ the additional contributions

$$\mathrm{i}\frac{\pi}{2}\left(f_{\rm i}^{\rm B}(0)-f_{\rm i}^{\rm A}(0)\right)=\mathrm{i}\frac{\pi}{2}\begin{cases}10/3 & (i=1,2),\\ 1/3 & (i=3),\\ 3 & (i=4),\\ 0 & (i=5,6)\end{cases} \tag{194}$$

follow which contribute to the imainary part of the form factors. In the remaining expressions of part A the rotation results in the substitution $\lambda\to-\lambda+{\rm i}0$ in the Euclidean formulas. The prescription $+{\rm i}0$ is necessary because the functions $m_{\rm i}(s,t)$, (188), have for $i=2,\dots,6$ a pole in $t=-1/2$. This pole delivers a further contribution to the imaginary part.

Finally we integrate by parts back and obtain

$$\begin{aligned}&\Pi^{(i){\rm ren}}(p) \\ &=-\ln B f_{\rm i}(0)+\mathrm{i}\frac{\pi}{2}\left(f_{\rm i}^{\rm B}(0)-f_{\rm i}^{\rm A}(0)\right)+\Pi_{(i)}^{\rm A}+\Pi_{(i)}^{\rm B},\end{aligned} \tag{195}$$

with

$$\begin{aligned}\Pi_{(i)}^{\rm A}&=\int_0^1\frac{d\lambda}{\lambda}\left(f_{\rm i}^{\rm A}(-\lambda)-f_{\rm i}^{\rm A}(0)\right)+\int_1^\infty\frac{d\lambda}{\lambda}f_{\rm i}^{\rm A}(-\lambda),\\ \Pi_{(i)}^{\rm B}&=\int_0^1\frac{d\lambda}{\lambda}\left(f_i^{\rm B}(\lambda)-f_i^{\rm B}(0)\right)+\int_1^\infty\frac{d\lambda}{\lambda}f_i^{\rm B}(\lambda){\rm e}^{{\rm i}\lambda m_g^2}.\end{aligned} \tag{196}$$

Now we have to deal with the infrared divergence for $m_g\to 0$. It appears only from $f_{\rm i}^{\rm B}(\lambda)$ because

$$f_{\rm i}^{\rm B}(\lambda)\to\frac{1}{2} \quad\text{for}\quad \lambda\to\infty \quad i=2,\dots,6 \tag{197}$$

holds. By adding and subtracting this asymptotic value we get

$$\begin{aligned}\Pi_{(i)}^{\rm B}&=\int_0^1\frac{d\lambda}{\lambda}\left(f_i^{\rm B}(\lambda)-f_i^{\rm B}(0)\right)+\frac{1}{2}\left(-\ln\frac{m_g^2}{B}+{\rm C}\right)\\ &\quad+\int_1^\infty\frac{d\lambda}{\lambda}\left(f_i^{\rm B}(\lambda)-\frac{1}{2}\right)+O\left(m_g^2\right)\\ &\equiv\frac{1}{2}\left(-\ln\frac{m_g^2}{B}+{\rm C}\right)+\tilde\Pi^{\rm B}\end{aligned} \tag{198}$$

for $i=2,\dots,6$, where C is Euler's constant. For $i=1$, $f_i^{\rm B}(\lambda)$ vanishes for $\lambda\to\infty$ and $\Pi_{(1)}^{\rm B}$ is given by (196) for $m_g=0$ directly in the same way as $\Pi_{(i)}^{\rm A}$ since the $f_{\rm i}^{\rm A}(-\lambda)$ vanish too.

Table 1. Results of numerical integration

i	Π^A	Π^B	$\tilde\Pi^B$
1	$0.012-0.52{\rm i}$	6.16	
2	$0.62+1.90{\rm i}$		-0.167
3	$0.32-0.11{\rm i}$		4.69
4	$0.18+0.11{\rm i}$		3.53
5	$0.16+0.16{\rm i}$		-0.97
6	$-0.16-0.16{\rm i}$		0.31

Now the integrations can be done and deliver numbers; see Table 1. In this way we get the correction

$$\begin{aligned}\Delta E_n^2&=\langle p_{||},n,s\,|_\alpha\,\Pi_{\alpha\beta}(p)\,|\,p_{||},n,s\rangle_\beta\\ &=\left(-\frac{1}{2}\left(-\ln\frac{m_g^2}{B}+{\rm C}\right)+1.807-7.15{\rm i}\right)\frac{g^2}{16\pi^2}B\end{aligned} \tag{199}$$

to the energy of the lowest level. Here we restored the dependence on all prefactors.

6 Conclusions

To summarize the results pesented in this paper we note that the gluon polarization tensor in a homogeneous magnetic background field is not transversal. As a consequence the polarization tensor has 10 independent tensor structures and, accordingly, 10 form factors. Seven of these tensor structures are transversal and three are not. We found this non-transversality from the explicit calulations in one-loop order. It shoud be mentioned that the non-transversality takes place in general. Only in special cases parts of the polarization tensor may be transversal. For example, this is known for the neutral component in Fujikawa gauge.

For the calculation of the form factors Schwinger's proper time method was used. The form factors are represented as double parameter integrals, the integrands have a form close to what is known from similar calculations in QED. They are even not much more complicated. In general, in this representation an integration by parts in the parameter integrals is desirable in order to get all contributions in the same representation. In the case of the neutral polarization tensor this is simple, for the charged polarization tensor it is not known how to do that. We solved this problem by making use of a sufficiently explicit representation we derived for the non-transversal part.

One of the opened problems is the definition of a suitable operator for the spin projection like $\sigma_{\mu\nu}$ in the spinor case. What can be done with the tensor structures found here is to investigate the level shifts and splitting in the magnetic field due to the radiation corrections given by the polarization tensor.

The tensor structures found in this paper may be used in two ways. First, they may serve as a input for the structures appearing in a Schwinger–Dyson equation in the magnetic background when the technique of the 2PI functionals is used. Second, they may be generalized to include temperature which should be a quite straight forward task now. Another interesting question is to investigate explicitly the dependence on the gauge parameter ξ of, for example, the level shift and splitting or the gluon condensate which should turn out to be gauge invariant. Using the methods developed here it should be possible to calculate the ξ-dependent part for the charged component too, which is however, beyond the scope of this paper.

The same structures can also be used if the W-boson mass operator in a magnetic field is investigated. This issue will be discussed elsewhere.

Another interesting question, of a technical character, is the integration by parts in the parameter integrals. For the neutral component it is known and serves as a check for the formulas obtained insofar as from the direct calculation of the non-transversal part the same result must appear. In the neutral case this is indeed the case. But for the charged component this check is missing.

Another task left for future work is to use the Slavnov–Taylor identities (as given, for example, in [14]) to derive the non-transversal parts and to compare with (117), (118), (169) and (170) which were derived from direct calculations.

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Appendix A

Here we note the explicit form of the operators (127) after rotation according to (27),

$$T^{(i)}_{\alpha\beta} = B^*_{\alpha\mu} T^{(i)}_{\mu\nu} B_{\nu\beta}. \tag{A1}$$

They read

$$T^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} l_4^2 & -l_3 l_4 \\ -l_3 l_4 & l_3^2 \end{pmatrix} \end{pmatrix},$$

$$T^{(2)} = \begin{pmatrix} \begin{pmatrix} n-1 & {a^\dagger}^2 \\ a^2 & n+2 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix},$$

$$T^{(3)} = \begin{pmatrix} l^2 & 0 & -\mathrm{i}a^\dagger l_3 & -\mathrm{i}a^\dagger l_4 \\ 0 & l^2 & \mathrm{i}al_3 & \mathrm{i}al_4 \\ \mathrm{i}al_3 & -\mathrm{i}a^\dagger l_3 & 2\hat{n}+1 & 0 \\ \mathrm{i}al_4 & -\mathrm{i}a^\dagger l_4 & 0 & 2\hat{n}+1 \end{pmatrix},$$

$$T^{(4)} = \begin{pmatrix} -l^2 & 0 & \mathrm{i}a^\dagger l_3 & \mathrm{i}a^\dagger l_4 \\ 0 & l^2 & \mathrm{i}al_3 & \mathrm{i}al_4 \\ -\mathrm{i}al_3 & -\mathrm{i}a^\dagger l_3 & -1 & 0 \\ -\mathrm{i}al_4 & -\mathrm{i}a^\dagger l_4 & 0 & -1 \end{pmatrix}, \tag{A2}$$

$$T^{(5)} = \begin{pmatrix} -l^2 & 0 & 0 & 0 \\ 0 & -l^2 & 0 & 0 \\ 0 & 0 & 2\hat{n}+1 & 0 \\ 0 & 0 & 0 & 2\hat{n}+1 \end{pmatrix},$$

$$T^{(6)} = \begin{pmatrix} -2(\hat{n}-1)-l^2 & 0 & 0 & 0 \\ 0 & 2(\hat{n}+2)+l^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The letter $\hat{n}$ in these formulas is the number operator, $\hat{n} = \frac{1}{2}(aa^\dagger + a^\dagger\, a)$.

The matrix elements of the tensors can be calculated easily. For $i = 1, 2, 3$ they follow from (129). The complete list for the transversal states reads

$$\begin{aligned} &\langle 1 \mid T_1 \mid 1\rangle = 0, && \langle 2 \mid T_1 \mid 2\rangle = \frac{l_4^2 h^2}{k^2}, \\ &\langle 1 \mid T_2 \mid 1\rangle = h^2, && \langle 2 \mid T_2 \mid 2\rangle = 0, \\ &\langle 1 \mid T_3 \mid 1\rangle = l^2, && \langle 2 \mid T_3 \mid 2\rangle = k^2 + \frac{l_3^2 l_4^2}{k^2}, \\ &\langle 1 \mid T_4 \mid 1\rangle = \frac{l^2}{h^2}, && \langle 2 \mid T_4 \mid 2\rangle = -\frac{k^4 + l_3^2 l_4^2}{h^2 k^2}, \\ &\langle 1 \mid T_5 \mid 1\rangle = -l^2, && \langle 2 \mid T_5 \mid 2\rangle = \frac{h^4 - l_3^3 l^2}{k^2}, \\ &\langle 1 \mid T_6 \mid 1\rangle = \frac{l^2 + 2h^2}{h^2}, && \langle 2 \mid T_6 \mid 2\rangle = \frac{h^4 - l_3^3 l_4^2}{k^2 h^2}. \end{aligned} \tag{A3}$$

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